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Interim Report
November 1979

ACTIVELY CONTROLLED STRUCTURES THEORY
Theory Of Design Methods

The Charles Stark Draper Laboratory, Inc.

Sponsored by
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ACTIVELY CONTROLLED STRUCTURES THEORY
Theory of Design Methods

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This is Volume 1 of the Charles Stark Draper Laboratory, Inc. Interim Technical Report on its Actively Controlled Structures Theory study. This portion of the report contains theoretical studies of selected constant gain output feedback controller design methods, and a performance comparison of specific designs with these methods against a common test example. On the basis of the theoretical studies and the performance comparisons, recommendations for focusing future research efforts are made.

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The Project Manager is Dr. Keto Soosaar and the Principal Investigator is Mr. Robert Strunce. This study was performed within the Advanced Systems Department headed by Mr. David Hoag. The contributors to Volume I of this report are as follows:

Dr. Jiguan G. Lin (Sections 1, 2, 4 and Appendices A, B);
Dr. Yu-Hwan Lin (Section 3 and Appendix C);
Dr. Daniel R. Hegg (Sections 6, 8);
Prof. Timothy L. Johnson (Section 7); and
Mr. James E. Keat (Section 5).

The authors gratefully acknowledge the assistance of Mr. Robert Strunce in performing the computer simulations.

The activities reported herein were conceived and initiated by the late Dr. Joseph Canavin who was, until his untimely death in an airplane accident in September 1978, the Principal Investigator of this program. He and his contributions are sorely missed by his collegues who respectfully dedicate this report to his memory.

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SECTION 1

GOALS AND MAJOR FINDINGS

1.1 Introduction

This interim report documents the Theory of Design Methods conducted in partial fulfillment of the Actively Controlled Structures Theory Study.* The objective of the studies in Volume 1 has been to analyze the theoretical aspects of selected constant-gain linear feedback design methods viewed as candidates for application to active vibration control of large space structures (LSS). Primary emphasis is placed upon:

(1) Research on constant-gain linear feedback methods applicable to active vibration control of large space structures.

(2) Evaluation of the suitability of this class of design methods to LSS controller design through application to a common test example.

(3) Identification of new extensions to these design methods which improve their applicability to LSS controller design.

(4) Identification of suitable directions for future efforts in developing theory and methods specifically applicable to the control of large flexible structures.

This volume contains preliminary studies on five of these design methods. The report on each of these methods includes: a brief introduction of the main ideas and underlying theory; an outline of the design method and/or the algorithm; a summary of assumptions made (or implied) and subtle techniques used; discussions on the strengths, weaknesses, maturity, and applicability to vibration control of large flexible space structures; illustration by a common test problem including answers to eight specific questions; and recommendations for improvement or further investigation.

1.2 Scope

The preliminary research conducted and presented in this volume concerns linear velocity output feedback methodologies:

(1) Modal Decoupling (Canavin).

(2) Pole Assignment (Davison-Wang).

(3) Optimal Output Feedback (Levine-Athans).

---

* Application of Design Methods is given in Volume 2.

(4) Suboptimal Output Feedback (Kosut)

(5) Stochastic Optimal Output Feedback (Johnson).

These design methodologies cover the state-of-the-art in linear output feedback.

Section 2 provides an overview of these methods in perspective, and the general framework used in reporting our preliminary results. Two different viewpoints are provided for visualizing the close relationships between the methods. The general framework consists of generic mathematical models of large flexible space structures, the general format for presentation, and an illustrative test example with eight points of interest to be addressed.

Section 3 concentrates on Canavin's method of modal decoupling. We examined the underlying theory, formulated those conditions which were implicitly assumed, completed a proof of a general stability theorem initially stated by Canavin, and provided another useful general stability theorem and its proof. Our evaluation is that this method has many desirable properties without major theoretical or computational problems.

Section 4 concentrates on Davison and Wang's method of pole assignment. We examined, analyzed, and integrated the scattered pieces of the underlying theory, clarified and organized the design procedure, uncovered a potential pitfall in closed-loop stability, discovered the insufficiency of using only velocity sensors in assigning all the desired closed-loop poles, studied previous applications of this method to attitude control of spacecraft with flexible appendages, and formulated several modifications for removing its serious theoretical and computational weaknesses. Our evaluation is that this method has many advantages and can be a very viable tool for preliminary or prototype design of active control systems for large flexible space structures, but it is not yet mature and requires extensive further research in order to be a feasible tool for LSS controller design.

Section 5 concentrates on Levine and Athans' method of optimal output feedback control. We examined the basic problems, considerations, and techniques for applying this method to vibration control of large flexible space structures, performed an extensive literature search on this method and previous applications of it, set up the rudiments of a technique for attacking the problems which result from the necessity of having a very good initial estimate of the controller gain matrix, suggested alternatives for reducing its computational difficulties, and showed by the test example that closed-loop stability is not guaranteed. Our evaluation is that computational difficulties in applying this method can overshadow its potential benefits.

Section 6 concentrates on Kosut's approximation of optimal output feedback control. We have carefully examined Kosut's two approximation methods and made significant theoretic extensions that make these methods applicable to arbitrary sensor configurations on large space structures. We demonstrated by the test example that the effects of control spillover can be significantly alleviated and that damping of the residual mode can be adjusted at will. Our evaluation is that these methods (in the currently published form) can

yield nearly the same optimal solutions as the Levine-Athans method, whereas the computational efforts required are significantly reduced. The extended versions can give significantly better performance.

Section 7 concentrates on Johnson's method of stochastic optimal output feedback control. We explored a stochastic formulation of the control problem with large flexible space structures for dealing with modal truncation and spillover (e.g., by treating control spillover as plant noise and observation spillover as measurement noise), developed an algorithm for stochastic optimal output feedback control, explored the idea of coupling residual modes with critical modes (by properly synthesizing measurement and control signals) to make the residual modes inherit some of the closed-loop stability properties of the critical modes, and demonstrated the feasibility of these new ideas. Our evaluation is that this method has many desirable features (some are unique) for application to active control of large flexible space structures, but it is too early to assess the complexity of the computational procedure at its current development stage.

Section 8 contains an overall comparison of the five methods reported using the numerical results obtained from applications of these methods to a common test problem. Recommendations for near-term research efforts are given.

Appendix A contains discussions on two common model-reduction approaches as applied to large flexible space structures, and on methods for comparing the relative importance of the vibration modes. A comment on the direct application of the conventional frequency-response method to undamped systems of harmonic oscillators is also given.

For direct regulation of modal responses of a large flexible space structure, or for alteration of its modes of response, the critical modes must be made completely controllable and completely observable if these modes are to be actively controlled. Appendix B presents necessary and sufficient conditions for selecting the location and number of actuators and sensors to guarantee complete controllability and observability. An algorithm is also presented.

Linear quadratic state-feedback regulators, as used in the (so-called) modern modal control systems, may not have robustness against modal truncation errors if they are used with Luenberger observers or Kalman filters. Appendix C summarizes recent results on the robustness of such regulators.

1.3 Recommendations

Based on the in-depth studies of the individual methods in Sections 3 through 7 and the performance comparisons for the test designs in Section 8, recommendations for near-term future research are briefly as follows:

(1) Discontinue further work on the Davison-Wang method for the present.

(2) Pursue specific theoretical studies to explore the possibility of extending the Canavin and the Levine-Athans methods so as to improve the design performance by exerting some influence over the residual modes.

(3) Continue theoretical development and initiate simulations of high-order systems using the Kosut and Johnson methods. This pairs a low-risk, moderate payoff approach with a high-risk, high-payoff approach.

SECTION 2

DESIGN METHODS FOR CONSTANT-GAIN FEEDBACK CONTROL SYSTEMS: INTRODUCTION AND OVERVIEW

2.1 Introduction

There are many modern methods for designing control systems for various applications. We chose the following seven methods to start our search for appropriate control strategies for large flexible space structures:

(1) State Feedback Control with a Luenberger Observer via Linear-Quadratic Regulation.

(2) State Feedback Control with a Luenberger Observer via Simon-Mitter Method of Pole Assignment.

(3) Output Feedback Control via Canavin Method of Modal Decoupling.

(4) Output Feedback Control via Davison-Wang Method of Pole Assignment.

(5) Optimal Output Feedback Control via Levine-Athans Method.

(6) Suboptimal Output Feedback Control via Kosut Approximation.

(7) Stochastic Optimal Output Feedback Control via Johnson Method.

This choice centers on the following theme: automatic constant-gain linear feedback control. The reasons are as follows. Linear automatic constant-gain feedback control systems as a class are much simpler to design, to implement, and to operate than other classes. If simple controllers designed by some of these methods are feasible and satisfactory for active vibration control of large flexible structures in space, why should one design and implement complex control systems: (1) that are not automatic and hence require constant human attention, or (2) that require on-line computation or generation of time-dependent gains, or (3) that require nonlinear control devices or schemes? On the other hand, understanding the weaknesses of simpler methods will offer useful insights as to what to look for in the search for suitable new methods.

Critical investigations into their underlying theories and their applicability to active vibration control of large flexible space structures are being conducted at CSDL on these methods. For the time being, however, we report only preliminary results on methods 3 through 7 (in Sections 3 through 7). Method 1 has become a standard approach to control system design; studies on its application to large flexible structures have been previously reported in References 6-7. Recent numerical experiments with a typical large flexible space structure are being covered in Volume 2 of this report, and a brief summary of recent developments on the sensitivity problem with this method is given in Appendix C of this Volume (also in Appendix B of Volume 2). Studies on method 2 will be continued and reported later.

Section 8 contains some preliminary conclusions from an overall comparison of Methods 3 through 7, and some preliminary recommendations. Comprehensive comparisons of all the seven methods will be conducted and reported later.

The purpose of the present section is to provide the reader an overview of these seven methods in perspective, and the general framework used in reporting our preliminary results. Two different viewpoints are provided in Section 2.3 for visualizing the close relationships between the seven methods. The general framework consists of generic mathematical models of large flexible space structures (Section 2.2), the general format for presentation (Section 2.4), and an illustrative test example with eight points of interest to be addressed (Section 2.5).

2.2 Mathematical Models of Large Flexible Space Structures

In order to minimize semantics and facilitate discussions of the various vibration suppression techniques for achieving modal control, the following model definitions are depicted in Figure 2-1. Large Space Structures are appropriately represented as distributed parameter systems (DPS) which require infinite-dimensional mathematical models. It is often more convenient to generate a physical model for a DPS by finite element methods; this results in a finite-dimensional modal representation. From this physical model, a reduced order evaluation model is selected such that the necessary model fidelity (a matter of engineering judgement) is maintained. Since this evaluation model may itself be sufficiently large so as to make the control of all modes infeasible, a design model is determined as a subset of the evaluation model. The design model must include those modes which degrade system performance beyond mission requirements.

The definitions of critical[†] modes, residual modes, observation spillover, and control spillover, as defined by Balas [6] are adopted.

> Critical modes, x_C, are those modes of the design model which are chosen to be explicitly controlled in order to assure stability and achieve performance requirements for the system.
>
> Residual modes, x_R, are those modes which exist in the infinite dimensional system that are not "critical" in the sense defined above.
>
> Observation spillover, is the contamination of the sensor outputs by the residual modes.
>
> Control spillover is the excitation of the residual mode dynamics due to the control.

Recently, some new design approaches to the vibration control problem have resulted in misinterpretations. This confusion has motivated an additional clarification. Residual modes can be subdivided into the following categories:

[†]We have adopted the term "critical", in place of the term "controlled" [6], to refer to those modes chosen to be explicitly controlled.

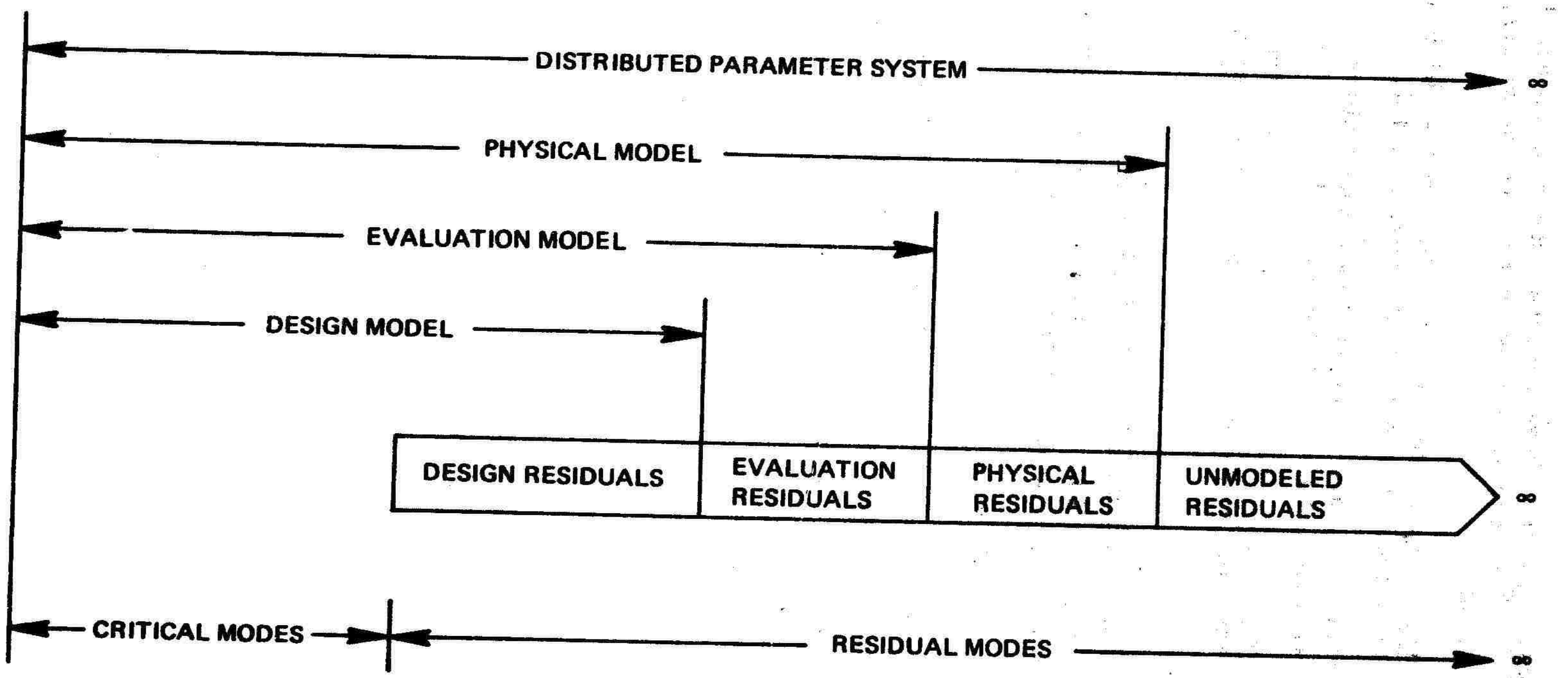


Figure 2-1. A pictorial representation of model definitions.

Design residual modes, x_{DR}, are those residual modes which are included in the controller design process (and thus in the design model), but are not explicitly controlled.

Evaluation residual modes, x_{ER}, are those residual modes which are included in the evaluation model but not in the design model.

Physical residual modes, x_{PR}, are those residual modes which are included in the physical model, but not in the evaluation model.

Unmodeled residual modes, x_{UR}, are those residual modes which exist, but are not included in any finite-dimensional model of the system under study.

2.2.1 Finite-Element Discrete Dynamic Model (Physical Model)

The dynamics of a large flexible space structure can be approximated by the finite-element method using a computer program like NASTRAN as follows:

$$M\ddot{q} + Kq = f \tag{2-1}$$

where $q \equiv (q_1, \ldots, q_L)$ is a vector of L generalized coordinates, $f \equiv (f_1, \ldots, f_L)$ is a vector of L generalized external forces, $M \equiv [M_{ij}]$ is a real symmetric positive definite matrix of L×L mass coefficients, $K \equiv [K_{ij}]$ is a real symmetric positive semi-definite matrix of L×L stiffness coefficients. The finite integer L is usually very large.

Forces or torques are applied through m actuators to control the structure:

$$f = B_A u \tag{2-2}$$

where $u \equiv (u_1, \ldots, u_m)$ denotes an m-vector of inputs to the actuators, and B_A is an L×m matrix of influence coefficients.

Observations are made through ℓ velocity or position sensors:

$$y = C_P q + C_V \dot{q} \tag{2-3}$$

where $y \equiv (y_1, \ldots, y_\ell)$ denotes an ℓ-vector of outputs from the sensors, C_P is an ℓ×L matrix of position coefficients and C_V is an ℓ×L matrix of velocity coefficients. If the ℓ sensors consist of ℓ_V velocity sensors and ℓ_P position sensors, separately located, the coefficient matrices C_P and C_V take the following special forms:

$$C_P = \begin{bmatrix} C_{P1} \\ \hline 0 \end{bmatrix} \begin{matrix} \}\ell_P \\ \}\ell_V \end{matrix} \qquad C_V = \begin{bmatrix} 0 \\ \hline C_{V2} \end{bmatrix} \begin{matrix} \}\ell_P \\ \}\ell_V \end{matrix}$$

Equations (2-1) and (2-3) constitute a finite-element discrete dynamic model of a generic large flexible space structure.

2.2.2 Finite-Element Modal Dynamic Model

In terms of vibration modes $\{\omega_j, \phi_j\}_{j=1}^{j=L}$ of the large flexible space structure, where ω_j denotes the natural frequencies (in rad/s) of vibration and ϕ_j denotes the mode shapes, the above discrete dynamic model can be rewritten as follows:

$$\ddot{\eta} + \Omega^2\eta = \Phi^T B_A u \tag{2-4}$$

$$y = C_P\Phi\eta + C_V\Phi\dot{\eta} \tag{2-5}$$

where $\eta \equiv (\eta_1,\ldots, \eta_L)$ is an L-vector of modal coordinates, $\Omega^2 \equiv \text{diag}\{\omega_1^2,\ldots, \omega_L^2\}$ is an L×L diagonal matrix of natural frequencies squared, and $\Phi \equiv [\phi_1,\ldots, \phi_L]$ is an L×L matrix of mode shapes. The superscript "T" denotes transpose. The matrix Φ possesses the following properties:

$$\Phi^T M\Phi = I, \quad \Phi^T K\Phi = \Omega^2.$$

The relationship between the generalized coordinates $(q_1,\ldots, q_L)$ and the modal coordinates $(\eta_1,\ldots, \eta_L)$ is given by the following vector equation:

$$q = \Phi\eta \tag{2-6}$$

2.2.3 Fundamental Modal Design Model

The order L of model (2-4) and (2-5) is in general too large for designing and implementing a control system on the large space structure. The order must be reduced to a practical level. On the other hand, the L vibration modes are not equally important. Some possible ways for determining the relative importance of the vibration modes are discussed in Appendix A. Figure 2-1 illustrates a useful partition of the L modes modeled by Eq. (2-4).

Let $\{\omega_{Cj}, \phi_{Cj}\}$, $j = 1,\ldots, N$, denote the critical modes, and $\{\omega_{Rk}, \phi_{Rk}\}$, $k = 1,\ldots, M$, denote the remainder of the L modeled modes, where N is the number of critical modes and $M \triangleq L - N$ is the number of modeled residual modes. Then the finite-element modal dynamic model (2-4) and (2-5) can be partitioned into critical and modeled residual parts as follows:

$$\ddot{\eta}_C + \Omega_C^2 \eta_C = \Phi_C^T B_A u \tag{2-7}$$

$$\ddot{\eta}_R + \Omega_R^2 \eta_R = \Phi_R^T B_A u \tag{2-8}$$

$$y = (C_P \Phi_C \eta_C + C_V \Phi_C \dot{\eta}_C) + (C_P \Phi_R \eta_R + C_V \Phi_R \dot{\eta}_R) \tag{2-9}$$

where

$$\eta_C = (\eta_{C1}, \ldots, \eta_{CN})$$

$$\Omega_C^2 = \text{diag}\,\{\omega_{C1}^2, \ldots, \omega_{CN}^2\}$$

$$\Phi_C = [\phi_{C1}, \ldots, \phi_{CN}]$$

$$\eta_R = (\eta_{R1}, \ldots, \eta_{RM})$$

$$\Omega_R^2 = \text{diag}\,\{\omega_{R1}^2, \ldots, \omega_{RM}^2\}$$

$$\Phi_R = [\phi_{R1}, \ldots, \phi_{RM}]$$

A most common approach in reducing a large mathematical model is to completely ignore the existence of those modes which are not dominant [1]. Following this approach, the large model (2-7) and (2-9) is simplified to the following fundamental design model which contains only critical modes:

$$\ddot{\eta}_C + \Omega_C^2 \eta_C = \Phi_C^T B_A u \tag{2-10}$$

$$y = C_P \Phi_C \eta_C + C_V \Phi_C \dot{\eta}_C \tag{2-11}$$

It is not really necessary to completely ignore all the non-dominant modes. The steady state of some residual modes may also be incorporated into the reduced design model to better approximate the steady-state response [2]. Some discussions on the reduced models thereby obtained are given in Appendix A. Nontheless, the reduced model (2-10) and (2-11) is the most common approach, and is appropriate for the purpose of studying these seven design methods.

2.2.4 Fundamental State-Space Design Model

Modern control methods, such as the seven design methods, are based on the following general state-space representation of the system to be controlled:

$$\dot{x} = Ax + Bu \tag{2-12}$$

$$y = Cx \tag{2-13}$$

where $x \equiv (x_1, \ldots, x_n)$ denotes a vector of n state variables, $u \equiv (u_1, \ldots, u_m)$ a vector of m control inputs, and $y \equiv (y_1, \ldots, y_\ell)$ a vector of ℓ observation outputs. A, B, and C are constant matrices of dimension $n \times n$, $n \times m$, and $\ell \times n$, respectively. Integers n, m, and ℓ denote the dimension of the state space, input space, and output space, respectively.

A convenient state-space representation of the fundamental modal design model (2-10) and (2-11) is given by

$$\dot{x} = A_C x + B_C u \tag{2-14}$$

$$y = C_C x \tag{2-15}$$

where

$$A_C = \begin{bmatrix} 0 & I \\ -\Omega_C^2 & 0 \end{bmatrix}, \quad B_C = \begin{bmatrix} 0 \\ \Phi_C^T B_A \end{bmatrix}, \quad C_C = [C_P \Phi_C, \; C_V \Phi_C].$$

2.2.5 Some Remarks on the Mathematical Models

The following are some important factors explicitly or implicitly affecting the applicability of the design methods to large flexible space structures.

2.2.5.1 Controllability and Observability of Critical Modes

The design of a feedback controller for the fundamental state-space design model (2-14) and (2-15), assumes that the critical modes are completely controllable and completely observable. Controllability and observability of critical modes, however, does not occur automatically. Improper location, or an improper number, of actuators used on the structure will make some critical modes uncontrollable. Similarly, improper location or an improper number of sensors used will also make some critical modes unobservable. See Appendix B for detailed discussions.

2.2.5.2 Rank of Matrices B_C, C_C

Some design methods also assume that the $n \times m$ matrix B of system (2-12) and (2-13) has rank m and the $\ell \times n$ matrix C has rank ℓ. (If B has rank n, then m-n actuators are redundant. Similarly, if C has rank n, then ℓ-n sensors are redundant.) For the $2N \times m$ matrix B_C of system (2-14) and (2-15) to have

rank m, it is necessary that $m \leq N$ because N rows are always zero. This implies that the number of independent actuators must not exceed the number of modes to be controlled. Otherwise, it is necessary to eliminate or combine some of the actuators. Similarly, for the $\ell \times 2N$ matrix C_C to have rank ℓ when all sensors are velocity sensors or all are position sensors, it is necessary that $\ell \leq N$. Otherwise, it is also necessary to eliminate or combine some of the sensors.

2.2.5.3 Control and Observation Spillover

In practice, controllers for a very large system, say model (2-7) through (2-9), are designed using its reduced-order model, say model (2-10) and (2-11). This essentially assumes, in the specific case of (2-7) through (2-9), that the matrices $\Phi_R^T B_A$, $C_P\Phi_R$, and $C_V\Phi_R$ are zero or negligibly small. However, such assumptions do not always hold for large flexible space structures.

If $\Phi_R^T B_A$ is not negligibly small, energy supplied for active control of fundamental modes may spill over to residual modes (unless vector u is such that $\Phi_R^T B_A u(t) \equiv 0$) and excite them according to the dynamics (2-8). The control of critical modes will not be affected by control spillover provided no sensor outputs are fed back, i.e., provided only open-loop control is applied.

On the other hand, if either $C_P\Phi_R$ or $C_V\Phi_R$ is not negligibly small, excited residual modes of vibration may "spill" over to the sensors and contaminate the observation of critical modes. Open-loop control of critical modes will again not be affected; feedback control of critical modes, however, will be affected, and the performance may become uncertain. Nonetheless, observation spillover without control spillover will not further excite the residual modes, and hence will not further degrade the feedback control of critical modes.

Simultaneous existence of both control and observation spillover is likely in large flexible space structures. Since feedback control of large flexible structures is considered, control and observation spillover combined may make the structure unstable. However, simultaneous existence of control and observation spillover need not be disastrous; it may be properly utilized to improve closed-loop stability (see Section 4.3.3., or Section 7).

2.3 An Overview of the Design Methods

2.3.1 Two Different Viewpoints

These methods can be looked at from two different viewpoints: state feedback vs. output feedback, and regulation of responses vs. alteration of modes of response. Since most are state-space methods, the following brief introduction will primarily refer to the general state-space representation (2-12) and (2-13).

2.3.1.1 State Feedback vs. Output Feedback

State feedback control of system (2-12) and (2-13) involves determining the control inputs $u_1, \ldots, u_m$ as functions of state variables $x_1, \ldots, x_n$:

$$u = Fx \tag{2-16}$$

where F is an m×n matrix of constant gains. Methods 1 through 3 compute the state-feedback gain matrix F in different ways.

Since state variables are usually not directly available for feedback purposes (such is particularly the case for large flexible space structures), an estimator must be used to provide the state-feedback controller with on-line estimates of the state from observation outputs $y_1 \ldots, y_\ell$. Methods 1 and 2 use a dynamic state estimator, namely, a Luenberger observer, while Method 3 uses a static state estimator. A state estimator can be considered as the dual of the corresponding state feedback controller. Thus, each method designs a state estimator in a similar but "dualized" manner as it designs a state-feedback controller. Method 1 usually bears the label of "modern control" though all other six methods are also modern control methods.

Output feedback control involves determining the control inputs $u_1, \ldots, u_m$ as functions of observation outputs $y_1, \ldots, y_\ell$

$$u = Gy \tag{2-17}$$

where G is an m×ℓ matrix of constant gains. Methods 3 through 7 compute the output-feedback gain matrix G in different ways.

The absence of dynamic state estimators is characteristic of Methods 3 through 7. Method 3 uses a static state estimator more for deriving a part of the gain matrix G than for actually estimating the state. These methods are extensions and modernizations of the classical concept of feedback controllers for single-input single-output linear time-invariant systems.

2.3.1.2 Regulation of Responses vs. Alteration of Modes of Response

Active control of structural vibration can be done by directly regulating the state vector whose components are modal responses $\eta_j(t)$ and their derivatives $\dot{\eta}_j(t)$. Methods 1, 5, 6, and 7 fall into this category. What these methods attempt to do is to minimize a quadratic performance index on the magnitude of the state vector. Methods 1, 5, and 6 also include the magnitude of control input in the quadratic performance index as a tradeoff between regulation accuracy achievable and control energy required. All these methods have essentially evolved from Kalman's contributions to the theory of optimal control, especially those dealing with linear-quadratic regulators [3].

Due to the questionable adequacy of summarizing the engineering specifications required of a large-scale system in a single quadratic

performance index, and the insurmountable computational problem with the solution of the associated matrix Riccati equation, Rosenbrock suggested the use of "modal control" as a design aid. Modal control of a multivariable system like (2-10) and (2-11), or system (2-12) and (2-13) in general, is by definition [4] to alter the modes of system response to achieve the desired control objectives. Since the modes of the system response are characterized by system poles, appropriate feedback (from state or output variables) is introduced to make the closed-loop system have the desired modes of response. Methods 2, 3, and 4 fall into this category. These methods have evolved mainly from Wonham's contribution to the theory of pole assignment [5].

2.3.2 Design Method 1: State Feedback Control with a Luenberger Observer via Linear-Quadratic Regulation.

The state-feedback control for system (2-12) and (2-13) is to be designed so that an infinite-time quadratic performance index

$$J = \int_0^\infty (x^TQx + u^TRu)dt$$

on the state vector x and input vector u is minimized. The approach is to obtain the state-feedback gain matrix F by solving the associated nonlinear n×n matrix algebraic equation of Riccati type. By shifting the eigenvalues of the system matrix A in the Riccati equation by a negative real number, the resultant closed-loop system will have all its poles lie to the left of this number. The stability margin is thus increased.

A Luenberger observer is used to reconstruct the state and hence provide the state-feedback controller with on-line estimates of the state. Since the dual of a Luenberger observer is also a state-feedback controller, the approach is to design the required observer by solving the corresponding Riccati equation.

This method has a sensitivity problem: the resultant closed-loop system is sensitive to model errors and parameter variations. This method also has the problem of choosing proper matrices Q and R to satisfy performance specifications.

2.3.3 Design Method 2: State Feedback Control with a Luenberger Observer via the Simon-Mitter Method of Pole Assignment.

The objective of state-feedback control of system (2-12) and (2-13) is to make the closed-loop system have desired modes of response. The approach is to compute the state-feedback gain matrix F using the Simon-Mitter or Crossley-Porter method of pole assignment. The gain matrix is a dyadic product of two vectors.

A Luenberger observer is used to provide the state-feedback controller with on-line estimates of the state. The approach is to design the required

observer by assigning desired closed-loop poles to the dual of system (2-12) and (2-13) again using the Simon-Mitter or Crossley-Porter method.

This method has the high-gain problem because of the imposed dyadic form of the gain matrix. The resultant closed-loop system is also sensitive to model errors and parameter variations.

2.3.4 Design Method 3: Output Feedback Control via the Canavin Method of Modal Decoupling.

This method is specifically developed for independently damping each critical mode of the underlying structure. It is an output-feedback method as well as a state-feedback method (with static state estimation). First of all, substituting (2-17) in (2-10) and (2-11) yields

$$\ddot{\eta}_C - \phi_C^T B_A G C_V \phi_C \dot{\eta}_C + (\Omega_C^2 - \phi_C^T B_A G C_P \phi_C)\eta_C = 0 \tag{2-18}$$

It is easily seen that negative feedback from velocity-sensor outputs will tend to add damping to the flexible structure. Canavin's method of modal decoupling uses only velocity sensors and computes a gain matrix G such that the resultant damping matrix $(-\phi_C^T B_A G C_V \phi_C)$ is diagonal and positive definite. In other words, Eq. (2-18) with $C_P = 0$ is to be decoupled in the modal coordinates as follows:

$$\ddot{\eta}_{Cj} + 2\zeta_{Cj}\,\omega_{Cj}\,\dot{\eta}_{Cj} + \omega_{Cj}^2\,\eta_{Cj} = 0$$

where ζ_{Cj} is the desired damping ratio on the jth critical mode.

The output-feedback control is a combination of state-feedback control

$$u = F\hat{\dot{\eta}}_C$$

and static estimation

$$\hat{\dot{\eta}}_C = Hy$$

where $\hat{\dot{\eta}}_C$ denotes an estimate of derivative $\dot{\eta}_C = (\dot{\eta}_{C1}, \ldots, \dot{\eta}_{CN})$. Therefore

$$G = FH$$

The approach is to compute the matrices F and H by solving the following matrix algebraic equations:

$$\phi_C^T B_A F = \text{diag}\,\{-2\zeta_{C1}\omega_{C1}, \ldots, -2\zeta_{CN}\omega_{CN}\}$$

$$C_V \phi_C H = I$$

This method requires a sufficiently large number of actuators and velocity sensors. The feedback gains may be high.

2.3.5 Design Method 4: Output Feedback Control via the Davison-Wang Method of Pole Assignment.

Conceptually, Canavin's method of output feedback control is a special method of pole assignment. It assigns N new complex-conjugate pairs of closed-loop poles to the fundamental state-space design model (2-14) and (2-15). The general method of pole assignment for system (2-12) and (2-13) does not require that the desired closed-loop poles preserve the same open-loop natural frequencies. The closed-loop poles can be freely chosen so as to meet other specified performance requirements.

In the Davison-Wang method of pole assignment, one systematically computes the output-feedback gain matrix G required for assigning as many as $\min\{m+\ell-1,n\}$ desired closed-loop poles to system (2-12) and (2-13). This method is different from the Simon-Mitter and Crossley-Porter methods, but the same technical approach is used: to compute the gain matrix G as a dyadic product of two vectors. This is achieved by converting a multivariable system either to a single-input system or to a single-output system. The computational procedure is therefore conceptually quite simple.

This specific method of pole assignment has its inherent weaknesses, such as high gains and hidden instability. The general method of pole assignment, if it eventually becomes practical, should not have such weaknesses, but would be much more complicated.

2.3.6 Design Method 5: Optimal Output Feedback Control via the Levine-Athans Method.

As with the linear-quadratic regulation of Method 1, the approach is to design an output feedback control so that the same quadratic performance index on state and control input is optimized. Uncertainty in the initial state is considered. No state estimators are required, however, since outputs are fed directly back to the system. In the method, one computes the feedback gain matrix G by recursively solving a linear $n \times n$ matrix algebraic equation of Lyapunov type and a nonlinear $n \times n$ matrix algebraic equation of Riccati type.

These matrix equations represent only first-order necessary conditions for optimality, and are highly coupled. Recursive computations are complex and their convergence is not guaranteed. Stability of the resultant closed-loop system is not guaranteed either.

2.3.7 Design Method 6: Suboptimal Output Feedback Control via the Kosut Approximation.

The same optimal output feedback control problem as in Method 5 is considered, but in this method one avoids the computational difficulties by seeking approximations of the optimal output feedback gains. Two different

ways of approximation are proposed: minimum error excitation and minimum norm.

In the minimum-error-excitation approximation, one minimizes a quadratic performance index on the error of the postulated output-feedback control inputs from the optimal. Only a linear n×n matrix algebraic equation of Lyapunov type needs to be solved for the suboptimal output-feedback gain matrix.

In the minimum-norm approximation, one minimizes the Euclidean norm of the error in the postulated feedback gain matrix from the optimal. The suboptimal output-feedback gain matrix is directly computable by matrix inversion and multiplication.

These approximately optimal output-feedback gain matrices are simple to compute, but the optimality of the design may be questionable. The stability of the resultant closed-loop system is not assured.

2.3.8 Design Method 7: Stochastic Optimal Output Feedback Control via the Johnson Method.

This method is still in the research and development stage. Gaussian white noise in the dynamics (2-12) and the observation (2-13) is considered. The output-feedback control optimizes an asymptotic mean-square measure of the state. This method in some respect is similar to Method 5, but it is more general and advanced in the sense that uncertainties in the dynamics and the observation (instead of only the uncertainty in the initial state) are taken into account. The output-feedback gain matrix G is to be obtained by recursively solving a linear n×n matrix algebraic equation of Lyapunov type and a nonlinear n×n matrix algebraic equation of Riccati type. The concept of "control projection" and "observation projection" is introduced to simplify the computations. Nonetheless, these matrix equations and recursive computations are much more complex than those of Method 5. An efficient numerical algorithm is currently being developed.

Its application to large flexible space structures is desirable, since the effects of control and observation spillover can be treated as stochastic disturbances instead of being ignored as usual. Furthermore, the method proposes that residual modes be coupled to critical modes by properly combining redundant actuators and sensors to make positive use of spillover so as to enhance closed-loop stability. Stability of the critical modes can then be inherited by the residual modes. Such an idea of utilizing control and observation spillover has been successfully tested on the two-mode mass-spring example (see Section 2.5). No general combination procedure is available yet. Such a procedure is desirable but is expected to be rather complex, since it will involve a large finite-element model (2-1) through (2-3) or (2-4) through (2-6).

2.4 General Format for Individual Reporting of the Design Methods.

The reporting of individual studies on Methods 3 through 7 uses the general format outlined in the following.

The general format consists of five main parts: Background, Discussions, Illustration, Conclusions, and References. Appendices may be included if necessary. The following is an outline of the four main parts.

Part I: Background

1. Brief introduction of the main ideas and underlying theory of the individual method.

2. Outline of the design method and/or the algorithm.

3. Summary of assumptions made and technical tricks used.

Part II: Discussions

1. Strengths.

2. Weaknesses (including theoretical limitations, numerical difficulties, and potential pitfalls).

3. Maturity (including improvements made or required).

4. Applicability to vibration control of large flexible space structures (including closed-loop stability, robustness to model errors and parameter variations, control and observation spillover, and special problems).

Part III: Illustration

Apply the individual method to the simple test problem given in Section 2.5 and address the eight points of interest listed therewith.

Part IV: Conclusions

1. Summary of advantages.

2. Summary of disadvantages.

3. Final comments (including recommendations for improvement or further investigation).

Part V: Appendices

(optional)

Part VI: References

2.5 A Simple Test Problem and Eight Points of Interest

The following example is simple and hand-calculable, but it is contrived (in the spirit of Reference 8) to capture many features and fundamental control problems of large flexible space structures.

Since the mass-spring system of the example was initially used by Henderson and Canavin in Reference 9 to demonstrate passive control by member dampers (i.e., dashpots) this example should serve as a good common basis for comparing various active control methods.

It is hoped that by working out this example with various methods, many useful theoretical and practical insights into the control of large flexible space structures can be generated.

An undamped mass-spring system whose axial (vertical) vibration is to be controlled is shown in Figure 2-2. The system has two modes: $f_1 = 0.087$ Hz, $f_2 = 0.412$ Hz. To capture fundamental features (and problems) in the control of large flexible space structures, imagine that this four-dimensional system is too large and that the controller design must be based only on a <u>reduced-order model</u>. Furthermore, only <u>modal control</u> is considered. Therefore, assume that Mode 2 is critical, and must be controlled so that it has at least 10% of critical damping (i.e., has damping ratio $\zeta \geq 0.1$). (In general, critical modes of a large space structure need not be ones with lower frequencies.)

The following is a summary of the data and the mathematical models useful as a convenient common reference.

Critical mode (to be controlled): $f_2 = 0.412$ Hz, $\phi_2 = \begin{bmatrix} -0.857 \\ 0.365 \end{bmatrix}$

$\omega_2 = 2.589$ rad/s

Residual mode (to be ignored): $f_1 = 0.087$ Hz, $\phi_1 = \begin{bmatrix} 0.516 \\ 0.606 \end{bmatrix}$

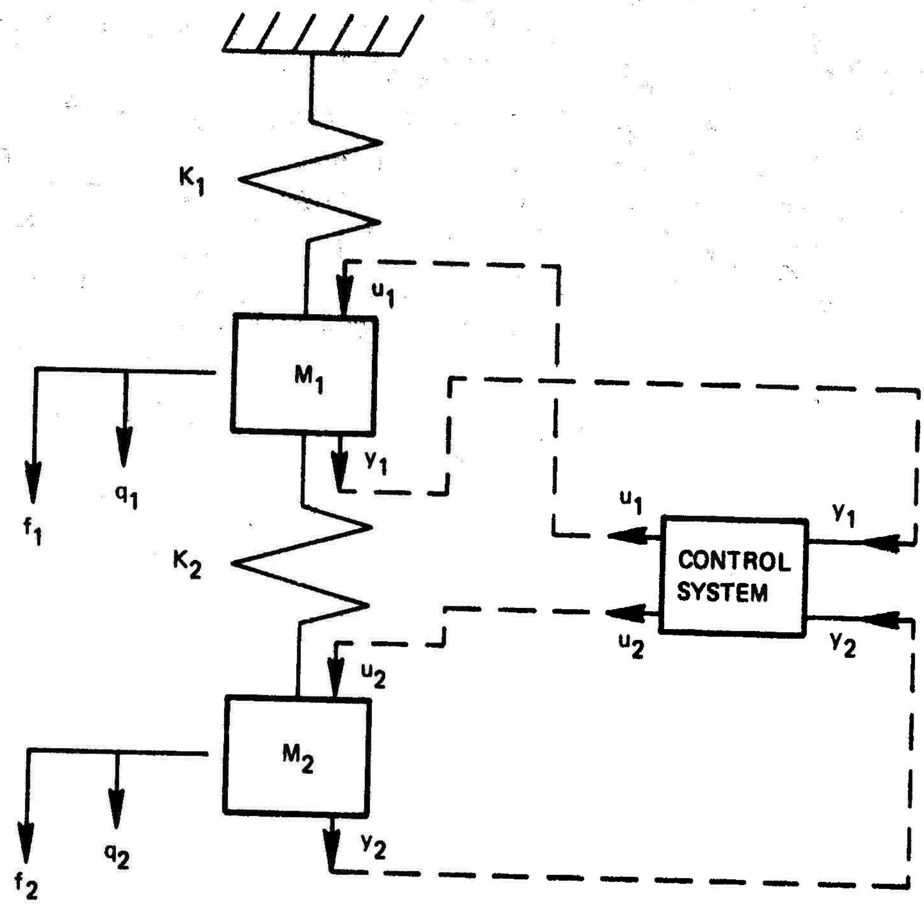
$\omega_1 = 0.546$ rad/s

<u>Finite-element discrete dynamic model</u>

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2-19}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \tag{2-20}$$

Plots of the open loop time responses to initial conditions and to a periodic disturbance, referred to both physical and modal coordinates, are shown in Figures 2-3 through 2-6.



M_1 = 1 Kg, M_2 = 2 Kg, K_1 = 1 N/m, K_2 = 4 N/m

q_i = displacement of mass i

y_i = output from velocity sensor i

u_i = input to force actuator i

Figure 2-2. A mass-spring system.

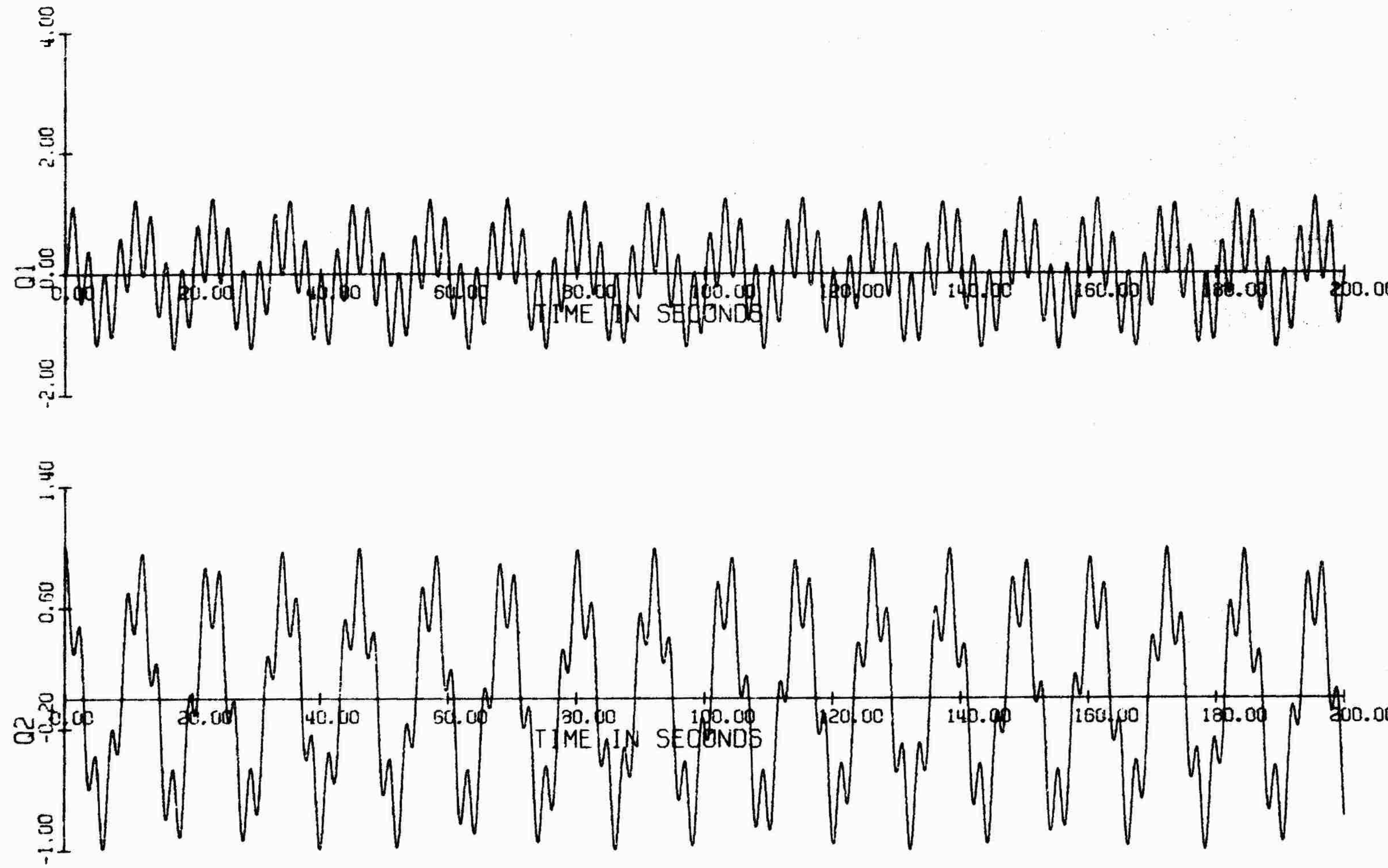


Figure 2-3. Open loop time responses $q_1(t)$ and $q_2(t)$ with $q_2(0) = 1$.

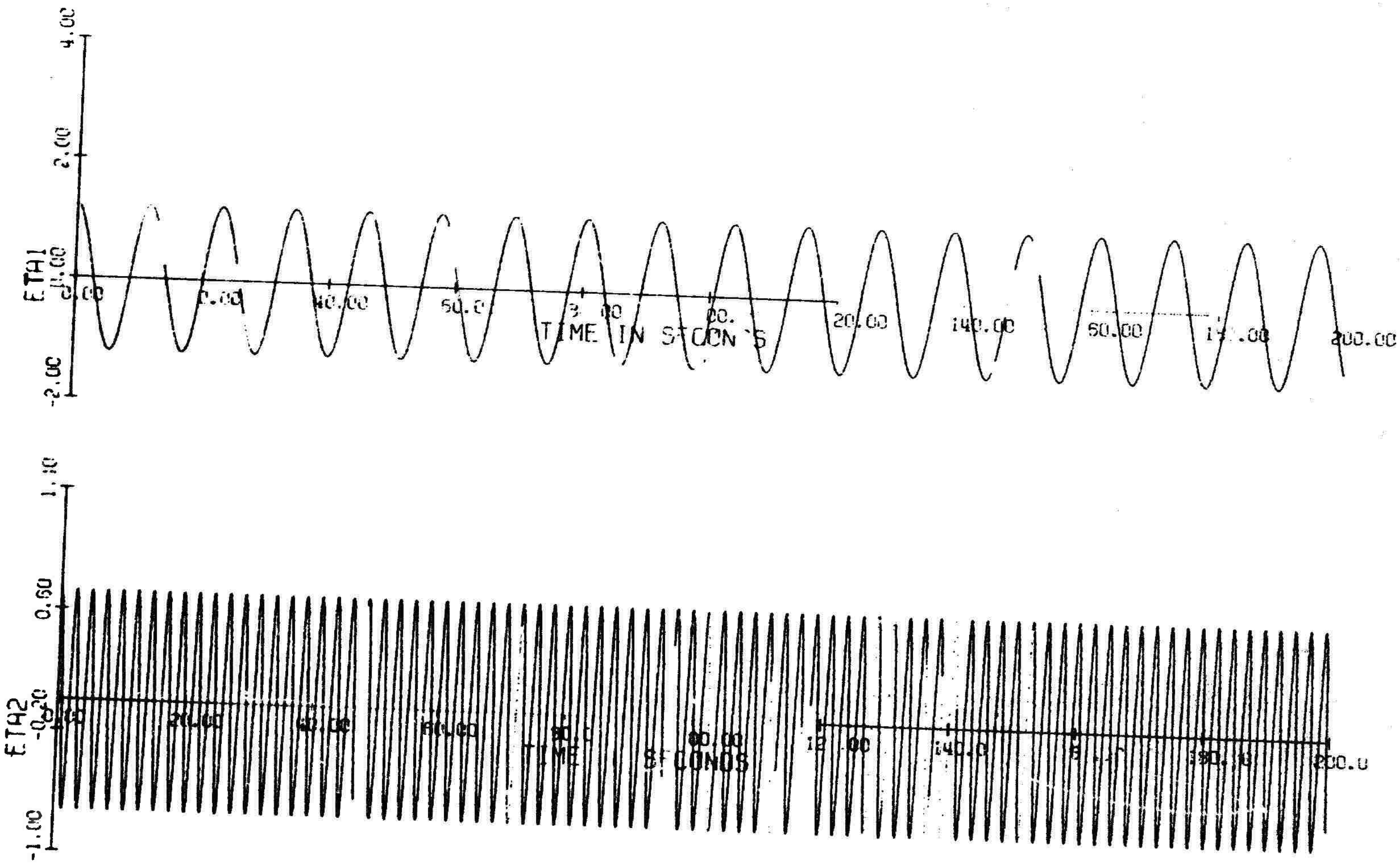


Figure 2-4. Open loop time responses $\eta_1(t)$ and $\eta_2(t)$ with $q_2(0) = 1$.

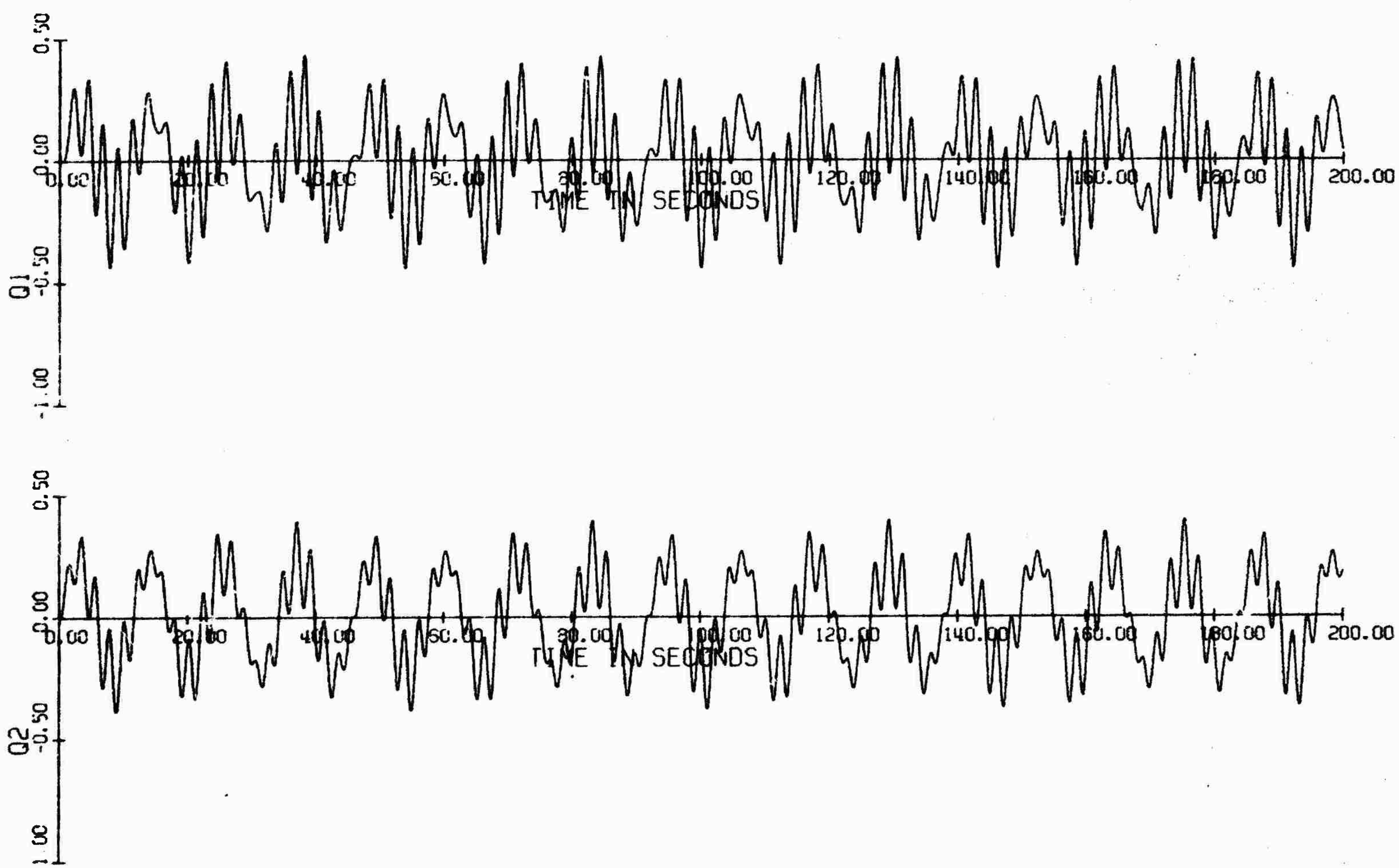


Figure 2-5. Open loop time responses $q_1(t)$ and $q_2(t)$ with $f_2(t)$ = sin 3t.

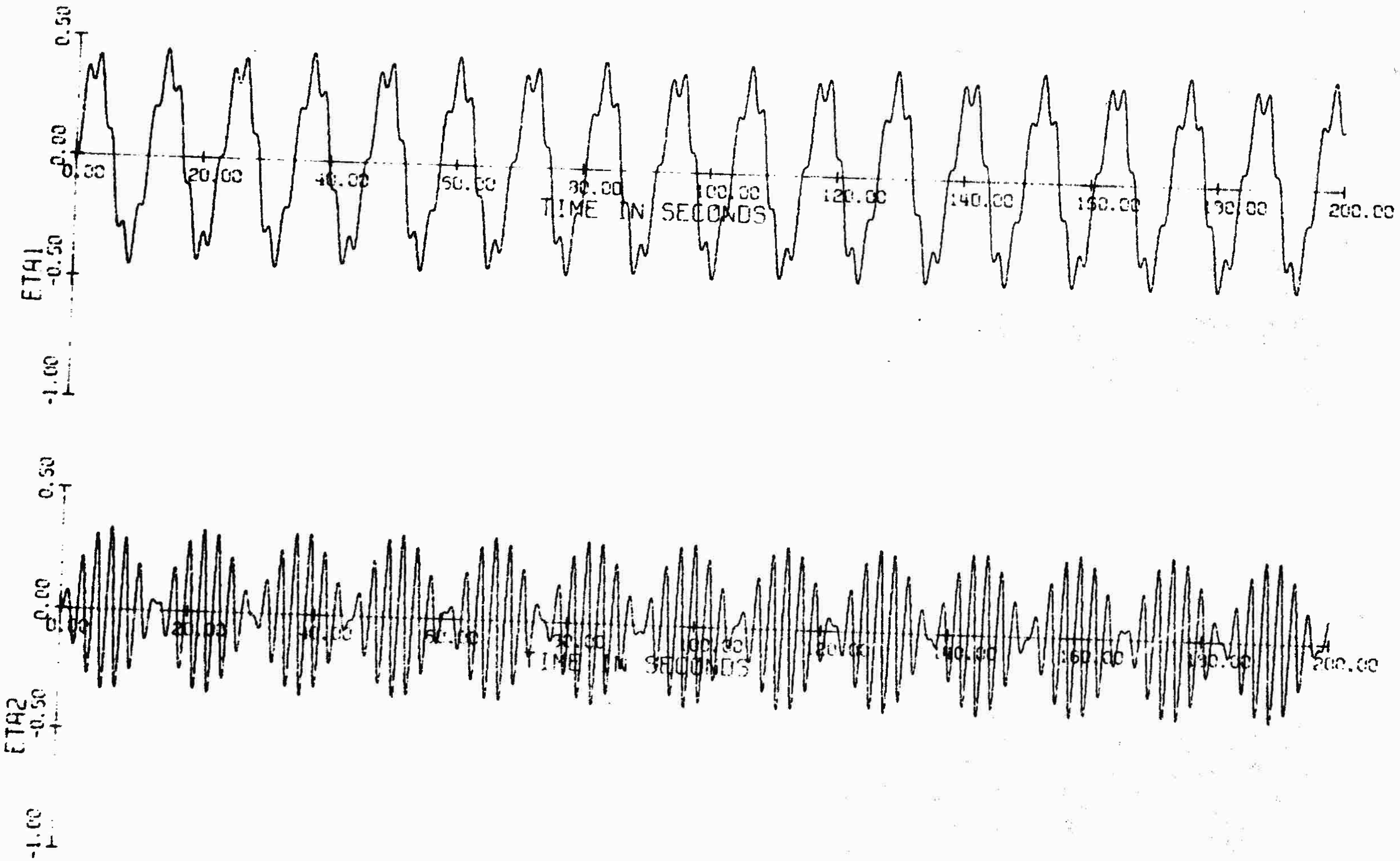


Figure 2-6. Open loop time responses $\eta_1(t)$ and $\eta_2(t)$ with $f_2(t) = \sin 3t$.

Finite-element modal dynamic model (Evaluation Model)

$$\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \phi^T \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2-21}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \phi \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} \tag{2-22}$$

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \phi \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}; \quad \phi = \left[\phi_1 \; \vdots \; \phi_2 \right]$$

Fundamental modal design model

$$\ddot{\eta}_2 + \omega_2^2 \eta_2 = \phi_2^T \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2-23}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \phi_2 \dot{\eta}_2 \tag{2-24}$$

Fundamental state-space design model

$$\begin{bmatrix} \dot{\eta}_2 \\ \ddot{\eta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_2^2 & 0 \end{bmatrix} \begin{bmatrix} \eta_2 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \phi_2^T \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2-25}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = [0 \quad \phi_2] \begin{bmatrix} \eta_2 \\ \dot{\eta}_2 \end{bmatrix} \tag{2-26}$$

First, design such a feedback controller using the specific method under study and the reduced-order model (2-23) and (2-24), or (2-25) and (2-26), having only the critical mode (f_2 = 0.412 Hz). One force actuator and one velocity sensor are to be attached to each mass, as shown in the figure, respectively, by u_i and y_i. Assume the actuators and sensors have no dynamics or noise. Then address the following points of interest.

1. Briefly demonstrate how the feedback gain matrix is computed.

2. With such a controller connected to the system, is the closed-loop system asymptotically stable? How much damping does each of the modes actually have?

3. What are the specific effects of control spillover and observation spillover with such a feedback controller? Note that Mode 1 is a residual mode.

4. Suppose mass 2 alone is subject to an initial disturbance: $q_1(0) = 0$, $q_2(0) = 1$ m, $\dot{q}_i(0) = 0$, $i = 1,2$. Describe the behavior of the closed-loop system, say in terms of the peak magnitude, 5% settling time, and steady-state response for $q_2(t)$. (The open-loop responses are shown in Figures 2-3 and 2-4 for comparison.)

5. Suppose mass 2 alone is subject to a persistent disturbance of $f_2(t) = \sin 3t$ Newton, with $q_i(0) = \dot{q}_i(0) = 0$, $i = 1,2$. How does the controller help suppress the vibration $q_2(t)$? Specifically, compare the steady-state response $q_{2ss}(t)$ of the closed-loop system with that of the original, open-loop system. (The open-loop responses are shown in Figures 2-5 and 2-6.)

6. Can the damping of all the modes (the residual as well as the critical) be further increased simultaneously by this method? How? Why? Is there any limitation?

7. Can the number of actuators and sensors be reduced? What is the minimum required by this method?

8. How would it affect the vibration control by this method if position (displacement) sensors were used instead?

References

1. Davison, E.J., "A Method for Simplifying Linear Dynamic Systems," IEEE Transactions on Automatic Control, Vol. AC-11, No. 1, pp. 93-101, January 1966.

2. Chidambara, M.R., "Further Remarks on Simplifying Linear Dynamic Systems," IEEE Transactions on Automatic Control, Vol. AC-12, pp. 213-214, April 1967.

3. Kalman, R.E., "Contributions to the Theory of Optimal Control," Bol. Soc. Mat. Mexicana, Second Ser., Vol. 5, pp. 102-114, 1960; Proceedings of the Mexico City Conference on Ordinary Differential Equations, 1959.

4. Simon, J.D., and S.K. Mitter, "A Theory of Modal Control," Information and Control, Vol. 13, pp. 316-353, October 1968.

5. Wonham, W.M., "On Pole Assignment in Multi-Input Controllable Linear Systems," IEEE Transactions on Automatic Control, Vol. AC-12, No. 6, pp. 660-665, December 1967.

6. Balas, M.J., "Active Control of Flexible Systems," Proceedings of AIAA Symposium on Dynamics and Control of Large Flexible Spacecraft, Blacksburg, VA, June 1977; also J. Optimization Theory and Applications, Vol. 25, No. 3, pp. 415-436, July 1968.

7. "Modern Modal Control," Passive and Active Suppression of Vibration Response in Precision Structures, State-of-the-Art Assessment, Volume 2: Technical Analysis, CSDL Report R-1138, February 1978, Section 4.

8. Canavin, J.R. "The Control of Spacecraft Vibration Using Multivariable Output Feedback, CSDL Report R-1137, January 1978; also Proceedings of AIAA/AAS Astrodynamics Conference, Palo Alto, CA, August 1978, Paper 78-1419.

9. "Member Dampers in Large Precision Structures," ibid., Section 2.

SECTION 3

OUTPUT FEEDBACK CONTROL VIA CANAVIN METHOD OF MODAL DECOUPLING

3.1 Background

3.1.1 Introduction

The decoupled-controller design technique [3] is an output feedback control scheme with no dynamic compensation and is used to achieve modal damping in a large space structure (LSS).

The decoupled-controller design technique presumes that the dynamics of large space structures can be represented by the finite-element model of Eq. (2-1). Futhermore, the modal dynamic model of Eq. (or system) (2-1) and its associated sensor output vector can be represented by Eqs. (2-7) through (2-9), which are repeated as follows:

$$\ddot{\eta}_C + \Omega_C^2 \eta_C = \Phi_C^T B_A u \tag{2-7}$$

$$\ddot{\eta}_R + \Omega_R^2 \eta_R = \Phi_R^T B_A u \tag{2-8}$$

$$y = (C_P \Phi_C \eta_C + C_V \Phi_C \dot{\eta}_C) + (C_P \Phi_R \eta_R + C_V \Phi_R \dot{\eta}_R) \tag{2-9}$$

where all symbols have been defined in Section 2.

The main idea of the decoupled-controller design is to choose a constant matrix G satisfying Eq. (2-17), i.e.,

$$u = Gy \tag{2-17}$$

such that Eq. (2-7) becomes

$$\ddot{\eta}_C + \Omega_C^2 \eta_C = -\left[\diagdown 2\zeta_C \Omega_C \diagdown\right] \dot{\eta}_C$$

where the diagonal matrix is

$$\left[\diagdown 2\zeta_C \Omega_C \diagdown\right] = \begin{bmatrix} 2\zeta_{C1}\omega_{C1} & & & 0 \\ & 2\zeta_{C2}\omega_{C2} & & \\ & & \ddots & \\ 0 & & & 2\zeta_{CN}\omega_{CN} \end{bmatrix}$$

If these ideas can be implemented, the dynamics of critical modes become decoupled and the amount of damping for each critical mode can be specified by choosing a desired value for ζ_C.

Canavin [3] proposed a method to implement these ideas and to guarantee the stability of the overall controlled structure. This section presents an evaluation of Canavin's method [3] with respect to the vibration control of large space structures.

The section is organized as follows: An outline of the design method and its assumptions is given in Section 3.1. A discussion of the design method is presented in Section 3.2. Section 3.3 presents the application of the method to a two mode example. It is ended with a conclusions/recommendation section.

The fundamental theories used in the development of the decoupled-controller design scheme include the generalized (or pseudo) inverse of a matrix [1] and the direct method of Liapunov [2]. Section 3.5 contains a summary of these theories to the extent necessary for the discussion of the decoupled-controller design in this section.

3.1.2 Outline of the Design Method (the Decoupled-Controller)

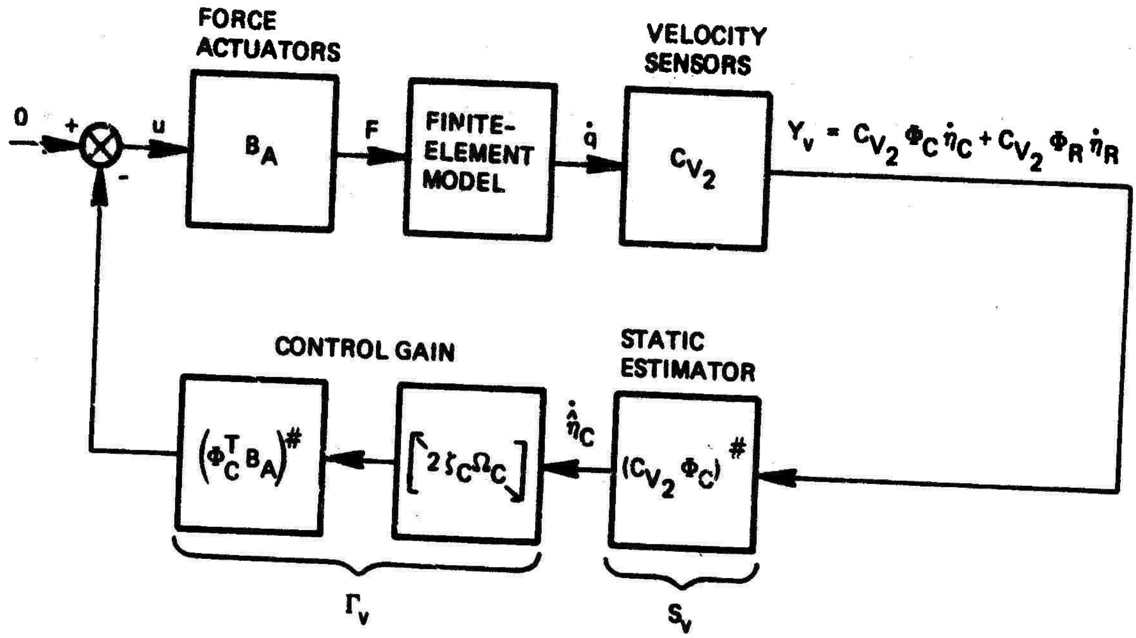
It is assumed that for the design of a decoupled-controller a design model in modal coordinates (i.e., Eq. (2-7)) is given and the required damping for each mode of the design model is also specified. The design method can then be summarized in the following steps:

(1) Choose only velocity sensors for measurements and force actuators for control. Form sensor/actuator pairs—each pair must be colocated. The number of sensor/actuator pairs must be greater than or equal to the number of modes in the design model.

(2) Refer to Figure 3-1. The constant output feedback gain is computed as the product of the static estimator gain (S_v) and the control gain (Γ_v):

a. The static estimator gain is equal to the pseudo inverse of $(C_{V_2}\Phi_C)$, i.e.,

$$S_v = \left(C_{V_2}\Phi_C\right)^{\#} \triangleq \left[\left(C_{V_2}\Phi_C\right)^T C_{V_2}\Phi_C\right]^{-1} \left(C_{V_2}\Phi_C\right)^T$$



$$(\Phi_C^T B_A)^{\#} \triangleq (\Phi_C^T B_A)^T \left[\Phi_C^T B_A (\Phi_C^T B_A)^T\right]^{-1}$$

$$(C_{V_2}\Phi_C)^{\#} \triangleq \left[(C_{V_2}\Phi_C)^T C_{V_2}\Phi_C\right]^{-1} (C_{V_2}\Phi_C)^T$$

$$B_A = C_{V_2}^T$$

Figure 3-1. The design method.

where C_{V_2} is the velocity output matrix defined in Section 2.2.1 and Φ_C is the matrix whose columns are the open-loop system eigenvectors corresponding to modes in the design model.

b. The control gain is equal to the product of two matrices, i.e.,

$$\Gamma_v = -(\Phi_C^T B_A)^{\#} \left[2\zeta_C\Omega_C\right]$$

where

$(\Phi_C^T B_A)^{\#} \triangleq (\Phi_C^T B_A)^T \left[\Phi_C^T B_A(\Phi_C^T B_A)^T\right]^{-1}$ and B_A is the influence matrix.

Moreover, $\left[\,^{\diagdown}2\zeta_C\Omega_{C\diagdown}\right]$ is a diagonal matrix and each diagonal element is chosen to be a constant $2\zeta_{Ci}\omega_{Ci}$, with ζ_{Ci} and ω_{Ci} representing the required damping ratio and the modal frequency, respectively, of the ith critical mode.

c. The constant output feedback gain G is then

$$G = \Gamma_V S_V$$

Ignoring the modal velocity of residual modes (i.e., $\dot{\eta}_R$) in the sensor outputs and considering only the design model (i.e., Eq. (2-7)), the governing equation for the critical modes, therefore, becomes

$$\ddot{\eta}_C + \Omega_C^2\eta_C = \Phi_C^T B_A u = \Phi_C^T B_A G y$$

$$\doteq \Phi_C^T B_A \Gamma_V S_V C_V \Phi_C \dot{\eta}_C = -\left[\,^{\diagdown}2\zeta_C\Omega_{C\diagdown}\right]\dot{\hat{\eta}}_C$$

where the estimated modal velocity $\dot{\hat{\eta}}_C$ is the output of a static least-square estimator (see Figure 3-1 and Section 3.5.2). That is

$$\dot{\hat{\eta}}_C = S_V C_V \Phi_C \dot{\eta}_C$$

Thus Canavin's method [3], due to the nonavailability of $\dot{\eta}_C$, uses $\dot{\hat{\eta}}_C$ to obtain damping. Also the decoupling mechanism, i.e.,

$$\Phi_C^T B_A \Gamma_V = -\left[\,^{\diagdown}2\zeta_C\Omega_{C\diagdown}\right]$$

is, in general, an approximate expression (see Section 3.5.2). Furthermore, sensor outputs often contain $\dot{\eta}_R$, and the actual damping ratio for each critical mode of the controlled structure is not in general identical to what is specified (i.e., ζ_{Ci}). Detailed discussions are given in Section 3.2.

3.1.3 Summary of Assumptions

To ensure that the design method outlined in Section 3.1.2 is a viable approach for structural vibration suppression, the following assumptions were made:

a. The stiffness matrix in the finite element model is symmetric and positive definite; i.e., rigid-body modes are not included in the dynamic finite element model. When the finite element model contains rigid-body modes, the stiffness matrix is symmetric and positive semi-definite. Then it is assumed that there is no strain energy with the rigid-body modes and that there are no control forces acting on these modes.

b. Sensor/actuator pairs can be colocated on the structure.

c. The number of sensor/actuator pairs is greater than or equal to that of controlled modes.

d. The design model and the required damping for each mode to be controlled are given.

3.1.4 Summary of Technical Techniques

Consider that the dynamics of the structure is represented by the design model. When the estimator gain is chosen as the pseudo inverse of $C_{V2}\Phi_C$, the output of the estimator is the best estimate ($\dot{\hat{\eta}}_C$) of the controlled model velocity ($\dot{\eta}_C$) vector in the least-square-error sense. The second pseudo inverse matrix $(\Phi_C^T B_A)^{\#}$ provides a decoupling mechanism with minimum control energy such that each controlled mode is damped by the feedback of its modal velocity as estimated by the estimator (see Section 3.5.2 for the meaning of these two pseudo inverse matrices). Furthermore, since only velocity sensors are used and sensor/actuator pairs are colocated, the system becomes energy dissipative, resulting in a stable overall system.

3.2 Discussion

3.2.1 Strengths

The decoupled-controller gain can be computed off-line. The computations involve only straightforward matrix multiplications and matrix inversions. Standard computer programs are available to carry out these tasks.

If only the design model is considered, the damping of the ith controlled mode (i.e., ζ_{Ci}) can be arbitrarily set in theory, by adjusting the corresponding values of the nonzero elements of the diagonal matrix $\left[{}^{\sim}2\zeta_C\Omega_{C\sim}\right]$.

The practical upper limits for achieving these damping ratios are determined by the power supply limitations and the saturation characteristics of the controller and actuators.

The decoupled-controller design also yields an overall stable system even in the presence of control and observation spillover. Furthermore, robustness of stability against model parameter errors or variations can also be demonstrated. These two properties (i.e., stability and robustness of stability) were originally obtained by Canavin [3] for structural vibration control where a general constant output feedback control design scheme is adopted, and the control force consists of both velocity and position output feedback, i.e.,

$$M\ddot{q} + Kq = f = -C_C\dot{q} - K_Cq \tag{3-1}$$

where the matrix C_C represents a general velocity feedback gain and K_C represents a general position feedback gain. It is noted that in the special case of the decoupled-control design, C_C and K_C take the following particular forms:

$$C_C \triangleq B_A\Gamma_v S_v C_{V_2}$$

$$= B_A(\Phi_C^TB_A)^T \left[\Phi_C^TB_A(\Phi_C^TB_A)^T\right]^{-1} \left[2\zeta_C\Omega_C\right] \left[(C_{V_2}\Phi_C)^T C_{V_2}\Phi_C\right]^{-1} (C_{V_2}\Phi_C)^T C_{V_2} \tag{3-2}$$

and

$$K_C \equiv 0 \tag{3-3}$$

In Section 3.5.4 the stability and the robustness of stability results are rederived, in a rigorous manner, for the general constant velocity and position output feedback control scheme so that some more insight can be obtained. However, these results can equally be applied to the special case of the decoupled-controller design, as specified by Eqs. (3-2) and (3-3).

A new stability result for the system (3-1) is given in Section 3.5.3, which is stronger than the theorem in Section 3.5.4. In essence, the new result (in Section 3.5.3) states that if M and $(K+K_C)$ are symmetric and positive definite and C_C is positive semi-definite, then the system (3-1) is asymptotically stable if and only if $M^{-1}(K+K_C)$ and C_C are an observable pair. This result is particularly useful in the application of the decoupled-controller design approach to structural vibration control, since in this case C_C is always positive semi-definite and asymptotic stability of system (3-1) is either desirable or required.

3.2.2 Weaknesses

Since the decoupled-controller design method allows an increase in the modal damping of each critical mode, the overall system performance requirements must be translated into modal damping requirements. If this cannot be done, control design iterations are required. Even when this can be done, the actual damping of the critical modes may vary when control and observation spillover are present. Furthermore, parameter errors or variations will change the degree of decoupling and estimation via Φ_C in the pseudo inverse of the matrices $(\Phi_C^T B_A)$ and $(C_V \Phi_C)$. Although stability may be retained, system performance may be degraded to an unacceptable level in the presence of parameter errors or variations.

Currently, the system stability requirement can be satisfied only if each sensor/actuator pair is colocated. This may impose unreasonable restrictions on the actual sensor/actuator placement due to physical constraints of the structure or practical implementation of the hardware.

Since the number of sensor/actuator pairs must be greater than or equal to that of the critical modes, it is expensive and complicated to implement the decoupled-controller with a large number of sensor/actuator pairs for structures that require a large number of modes to be controlled. In addition, the chance of hardware failure may become significant and its impact on the use of the decoupled-controller design may have to be reassessed.

3.2.3 Maturity

In 1976, Quartararo [4] proposed a modal control concept that is based on two coordinate transformations. The first coordinate transformation is responsible for transforming from the discrete coordinate to the modal coordinate, identical to the process from Eq. (2-1) to Eq. (2-4) as discussed previously. The second coordinate transformation is specifically introduced to achieve independent actuation of the modal equations and actuators are used in such a way as to produce a generalized force in any given mode without forcing the other modes [4]. Instead of using pseudo inverse matrices, Quartararo considered the case where the number of actuators equals the number of the critical modes and hence only a normal matrix inversion is needed. Similarly, a normal matrix inversion is used for state estimation because the number of sensors is equal to the number of critical modes.

Canavin [3] extended Quartararo's concept to include the use of pseudo inverse for matrices (see Background section) and obtained the stability and the robustness of stability results (see Section 3.5.4). Moreover, Canavin [3] proposed the velocity-only feedback design scheme as a special case that satisfied the stability criteria and thus provided an approach to achieve desired modal damping, stability and robustness of stability. However, large values in the velocity feedback gain matrix were reported in a numerical example to achieve ten percent of critical damping for the critical modes [3].

3.2.4 Applicability to Vibration Control of LSS

Theoretically, the decoupled controller design technique is suitable for LSS vibration control applications. As discussed in Section 3.2.1, a decoupled-controller with only velocity feedback yields a stable closed-loop system, which is robust against model error. Control and observation spill-over may degrade the system performance but cannot destabilize the system.

Associated with the decoupled-controller design approach, however, there remain issues to be resolved before its practical applicability to LSS vibration control can be completely assessed. These issues will be discussed in the Conclusions given in Section 3.4.

3.3 Illustration (The Example)

The model for the example system is repeated in the following for convenience:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 5 & -4 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = f \qquad (3\text{-}4)$$

Therefore, the mass matrix M and the stiffness matrix K take the form:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad K = \begin{bmatrix} 5 & -4 \\ -4 & 4 \end{bmatrix}$$

Two velocity sensors and two actuators are employed in this system in such a manner that the measurement y_v takes the form

$$y_v = C_{V_2} \dot{q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \qquad (3\text{-}5)$$

and the control force takes the form

$$f = B_A u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad (3\text{-}6)$$

It is noted that

$$C_{V_2} = B_A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and that

$$\Phi_C = \Phi_2 = \begin{bmatrix} -0.857 \\ \\ 0.365 \end{bmatrix}, \quad \text{and } \Phi_R = \Phi_1 = \begin{bmatrix} 0.516 \\ \\ 0.606 \end{bmatrix}$$

Furthermore,

$$\omega_1 = 0.5465 \text{ rad/s and } \omega_2 = 2.59 \text{ rad/s}$$

The eight questions posed in Section 2.5 are answered in the sequel.

3.3.1. Gain Computations

$$\Gamma_v = (\Phi_C^T B_A)^T \left[\Phi_C^T B_A (\Phi_C^T B_A)^T\right]^{-1} \cdot (2\zeta_2\omega_2) = \begin{bmatrix} -0.98769814 \\ \\ 0.42066491 \end{bmatrix} \cdot (2\zeta_2\omega_2)$$

$$S_v = \left[(C_{V_2}\Phi_C)^T (C_{V_2}\Phi_C)\right]^{-1} (C_{V_2}\Phi_C)^T = [(-0.98769814)(0.42066491)]$$

where ζ_2 is the damping ratio for the second (i.e., critical) mode. In theory, ζ_2 can be set to an arbitrary value. However, in this illustration ζ_2 is set to 0.1.

3.3.2 Stability and Actual Damping

First consider the design model with control. The closed-loop equation for the design model is then

$$\ddot{\eta}_2 + (\Phi_2^T B_A \Gamma_v S_v C_{V_2} \Phi_2)\dot{\eta}_2 + (\omega_2)^2 \eta_2 = 0$$

where

$$\Phi_2^T B_A \Gamma_v S_v C_{V_2} \Phi_2 = [(-0.875)(0.365)] \cdot (2\zeta_C^2 \omega_2)$$

$$\cdot \begin{bmatrix} (\ 0.97554762)(-0.41548995) \\ \\ (-0.41548995)(\ 0.17695397) \end{bmatrix} \begin{bmatrix} -0.857 \\ \\ 0.365 \end{bmatrix}$$

$$= 2\zeta_2\omega_2 = 2 \cdot (0.1) \cdot (2.58867) = 0.517734$$

It is apparent that if the design model is considered, the closed-loop system is asymptotically stable and the damping is equal to 10 percent of critical.

Next, consider the evaluation model with control. The closed-loop system equations become

$$\left\{\begin{array}{lr} \ddot{\eta}_2 + 2\zeta_2\omega_2\dot{\eta}_2 + (\omega_2)^2\eta_2 + \underbrace{(2\zeta_2\omega_2) \cdot (-0.2511)\dot{\eta}_1}_{\text{OBSERVATION SPILLOVER}} = 0 & (3\text{-}7) \\ \ddot{\eta}_1 + \underbrace{(2\zeta_2\omega_2) \cdot (0.06307)\dot{\eta}_1}_{\text{CONTROL \& OBSERVATION SPILLOVER}} + (\omega_1)^2\eta_1 + \underbrace{(2\zeta_2\omega_2) \cdot (-0.2511)\dot{\eta}_2}_{\text{CONTROL SPILLOVER}} = 0 & (3\text{-}8) \end{array}\right.$$

With ζ_2 chosen to be 0.1 and therefore $2\zeta_2\omega_2$ equal to 0.517734, the actual damping for η_1 is found to be 0.0306, and the actual damping for η_2 is 0.10006. It is noted that the damping for η_1 is positive and the damping for η_2 is only slightly different from the chosen value (i.e., 0.1).

The stability of the system (3-7) and (3-8) is indicated*, because the poles of the system have non-positive real parts:

$$P_1, P_2 = -0.016751 \pm j0.546779 \qquad (3\text{-}9)$$

$$P_3, P_4 = -0.258690 \pm j2.572255 \qquad (3\text{-}10)$$

3.3.3 Control and Observation Spillover Effects

Consider Eqs. (3-7) and (3-8) in which control and observation spillover are identified. One of the observation spillover effects is to couple the dynamics of η_2 with that of $\dot{\eta}_1$ (see Eq. (3-7)). Moreover, one of the control spillover effects is to drive the dynamics of η_1 with $\dot{\eta}_2$ (see Eq (3-8)). The combined effects of control and observation spillover, however, introduce a positive damping term in Eq. (3-8) which is partially responsible for having nonpositive real parts of the system poles of Eqs. (3-7) and (3-8).

3.3.4 The Behavior of the Controlled System with an Initial Disturbance

The dynamics of the controlled system can be represented by the following equation:

---

* For this example, all conditions required for the theorem in Section 3.5.3 to hold are satisfied and therefore asymptotic stability is in fact assured.

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 5 & -4 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = f = - \begin{bmatrix} 0.50507463 & -0.21511347 \\ -0.21511347 & 0.09161776 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \tag{3-11}$$

Solutions (i.e., $q_1(t)$ and $q_2(t)$) to Eq. (3-11) are obtained via simulation and are plotted against time in Figure 3-2. From this result, it is seen that q_2 is initially displaced by 1 meter which is its maximum displacement. In general, q_2 exhibits a behavior of decaying oscillations. Similarly, q_1 is initially at rest and subsequently also exhibits a behavior of decaying oscillations. However, the first peak displacement of q_1 is the maximum displacement of q_1 throughout the simulation period. It is noted that system poles of Eq. (3-10) correspond to a faster decaying rate than that of Eq. (3-9). Therfore, both q_1 and q_2 of Figure 3-2 exhibit an oscillation frequency of about 0.547 rad/s. From Eq. (3-9), the dominant time constant of the system is approximately 60 seconds. Therfore, the 5% settling time of the system is about 300 seconds. However, at steady-state, both q_1 and q_2 should approach zero since the closed-loop poles of the system (see Eqs. (3-9) and (3-10)) have negative real parts.

3.3.5 System Performance under Persistent Disturbance

System performance under a persistent sinusoidal disturbance is simulated with and without the decoupled-controller. The time-history of q_1 and q_2 in either case are plotted in Figure 3-3 and Figure 3-4. In the case where no control is employed, both q_1 and q_2 exhibit vibratory motions of no damping. The vibrations have three frequency components, i.e., the system natural frequencies and the frequency of disturbance. The magnitude of vibrations could reach about 0.4 meters for both q_1 and q_2. However, in the controlled case, the magnitude of the vibrations is reduced in both transient and steady-states. In particular, at steady-state the magnitude of the suppressed vibration is about 0.1 meter and the vibration consists only one frequency, i.e., that of the disturbance.

3.3.6 Simultaneous Increase in Damping of Both Modes

The closed-loop system equations (3-7) and (3-8) can be expressed in vector form:

$$\begin{bmatrix} \ddot{\eta}_2 \\ \ddot{\eta}_1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \dot{\eta}_2 \\ \dot{\eta}_1 \end{bmatrix} + \begin{bmatrix} \omega_2^2 & 0 \\ 0 & \omega_1^2 \end{bmatrix} \begin{bmatrix} \eta_2 \\ \eta_1 \end{bmatrix} = 0 \tag{3-12}$$

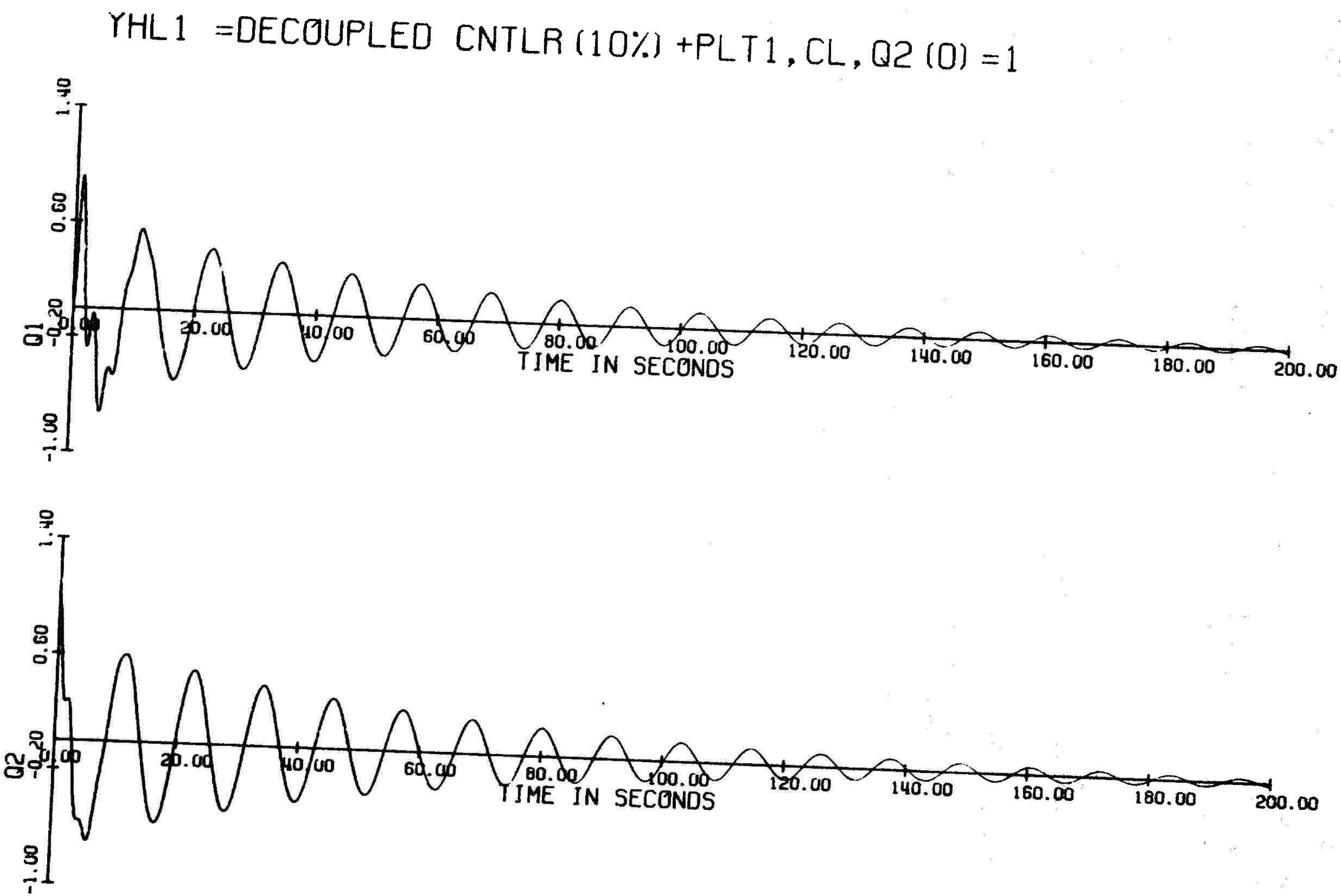


Figure 3-2. System performance with an initial disturbance.

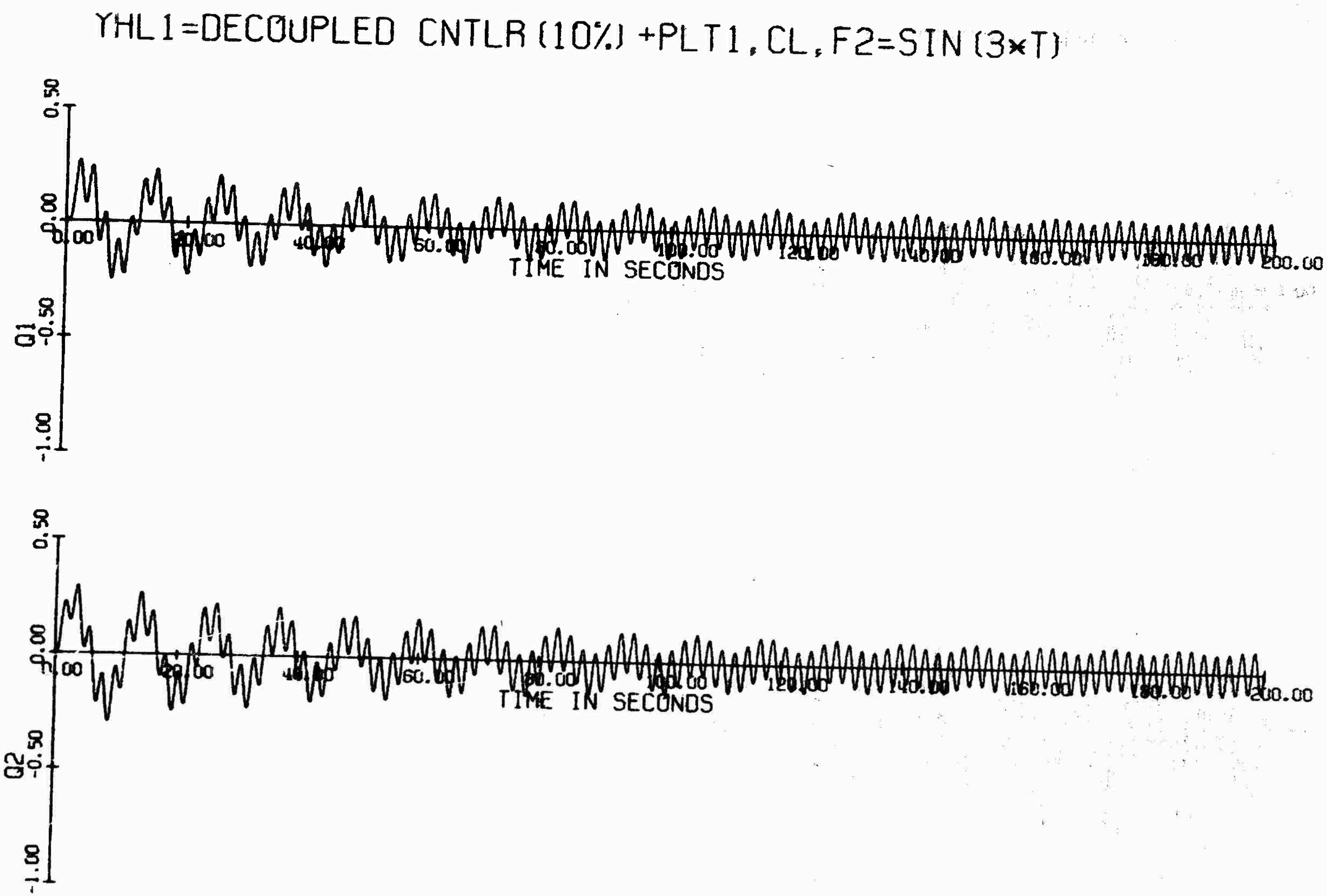


Figure 3-3. Controlled system under persistent disturbance.

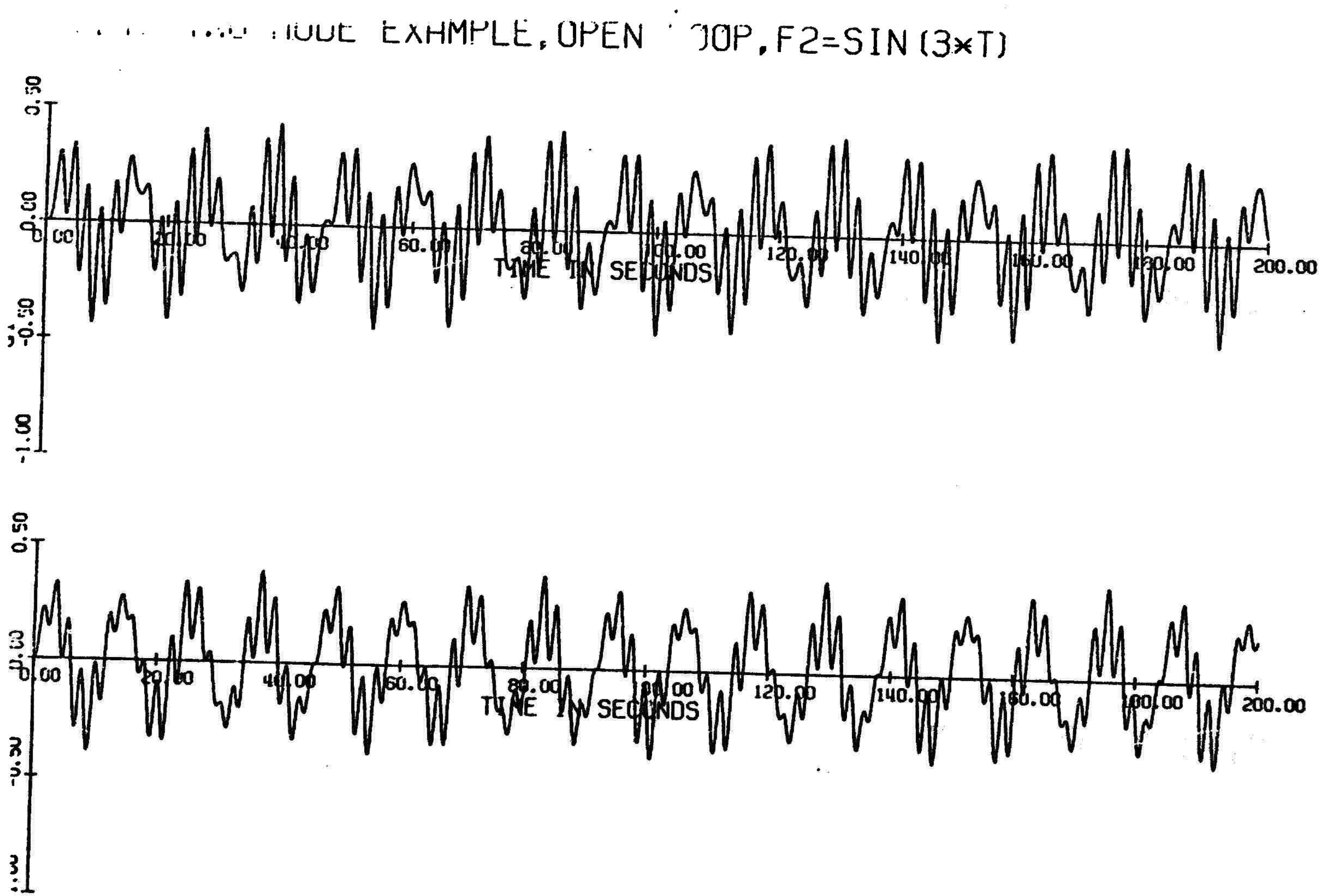


Figure 3-4. System without control under persistent disturbance.

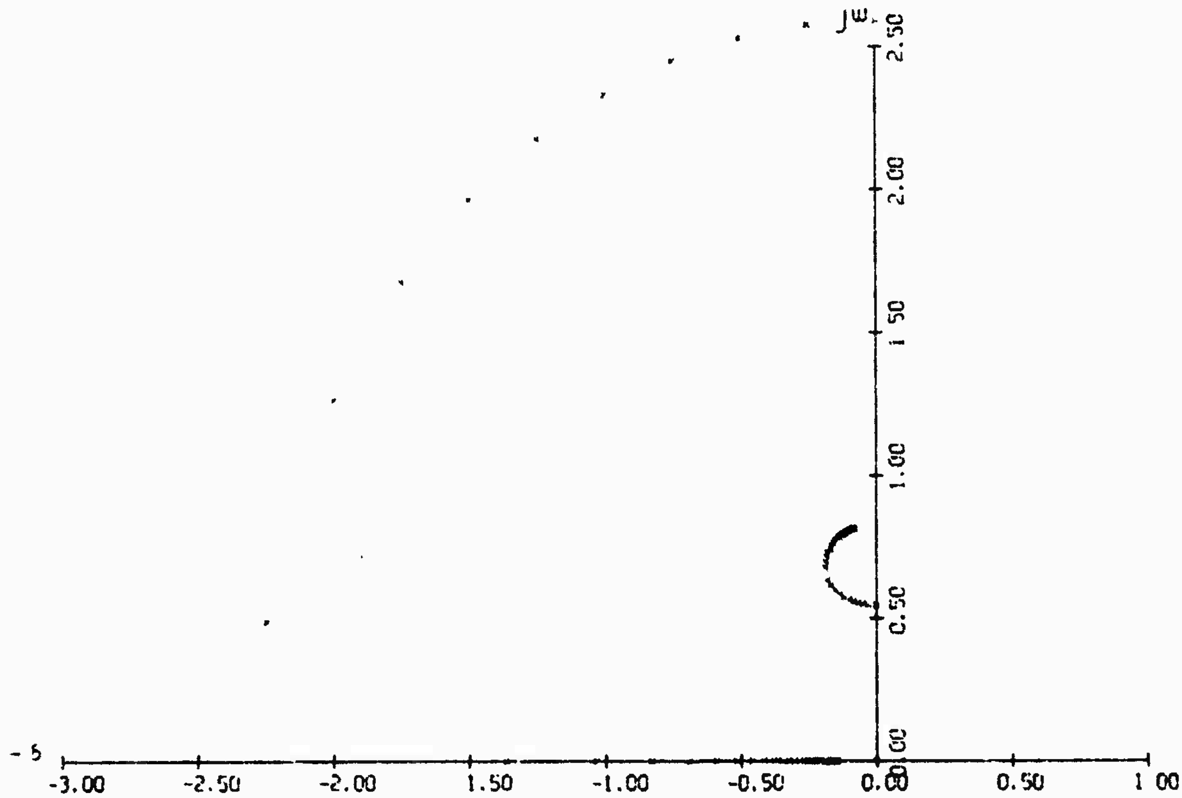
where

$$a = 2\zeta_2\omega_2$$

$$b = c = 2\zeta_2\omega_2 \cdot (-0.2511)$$

$$d = 2\zeta_2\omega_2 \cdot (0.06307)$$

and ζ_2 is the only free design parameter. The actual damping of each mode is a function of ζ_2. Numerical plots of the poles of Eq. (3-12) for several values of ζ_2 are given in Figure 3.5, which clearly shows that the damping of mode 2 (the critical mode) can theoretically be increased without bound by increasing the value of ζ_2. However, the achievable damping of mode 1 (the residual mode) has a finite maximum value of about 0.3 and after reaching this maximum damping the damping of mode 1 drops in spite of the increase in ζ_2. Nevertheless, the system (3-12) can be proved by the Routh-Hurwitz theorem to be asymptotically stable as long as $0 < \zeta_2 < \infty$.



(Contributed by Capt. D.C. Herrick)

Figure 3-5. Root locus of the closed loop example system as a function of the design parameter ζ_2.

3.3.7. Reduction in the Number of Sensor/Actuator Pairs

To take the decoupled-controller design approach, the minimum number of sensor/actuator pairs is one in order to design a controller for a design model with one mode. The design procedure in this case would have been similar to that discussed in the two mode example except that regular matrix inversion, rather than pseudo inverse of matrices, would be used.

3.3.8. Position Sensors

If only position sensors are used, the closed-loop system equation can be expressed in discrete (physical) coordinates as follows

$$M\ddot{q} + (K+K_C)q = 0 \tag{3-13}$$

where K_C is the position feedback matrix. If K_C is symmetric, Eq. (3-13) represents a system of oscillatory nature and the frequencies of the controlled modes may be changed.

3.4 Conclusions/Recommendations

3.4.1 Summary of Advantages

(1) Control gain computations are easy and straightforward; computation software is available.

(2) Damping of each critical mode can be arbitrarily set.

(3) Stability is guaranteed.

(4) Stability is retained in the presence of model errors.

3.4.2 Summary of Disadvantages

(1) System performance requirements must be translated into modal damping requirements.

(2) Modal damping requirements may not be satisfied in the presence of model errors.

(3) Sensor/actuator pairs must be colocated.

(4) The number of sensor/actuator pairs must be greater than or equal to the number of modes in the design model.

3.4.3 Recommendations

In light of the disadvantages of the decoupled-controller design approach, areas for further research include:

a. To relax the requirement of sensor/actuator colocation—this requirement was established in [3] to guarantee the positive semi-definiteness of C_C (see Eq. (3-2)), and therefore the stability of the system. However, the choice of $B_A = C_{V_2}^T$ is only a sufficient condition to guarantee such a C_C matrix. A more general condition may relax the requirement of sensor/actuator colocation (i.e., $B_A = C_{V_2}^T$).

b. To reduce the number of sensor/actuator pairs—the relationship between the number of sensor/actuator pairs and the system performance was not established in Reference [3]. The number of sensor/actuator pairs may be reduced to less than the number of critical modes at the expense of a dynamic feedback controller. The trade-off between the number of sensor/actuator pairs and the system performance should be investigated.

c. To perform sensitivity analysis—in the presence of model parameter errors or sensor/actuator failure, the actual damping of the critical modes may be different from the desired value. the extent of resulting system performance degradation should be determined.

d. To evaluate practical applicability of the method—in Reference [3], it was found that high gain was required when the decoupled-controller design approach was applied to a large space structure to achieve about 10% of critical damping for 12 critical modes. It would be desirable to determine if this high gain result was caused by the approach itself or by the placement of sensors and actuators. The sensor and actuator dynamics were completely ignored in Reference [3]. But in practical applications, sensor and actuator dynamics should be considered and therefore the overall system must be reevaluated.

3.4.4 Final Remarks

Since complex large space structures are considered, finite-element models of LSS may often be used. If constant output feedback control techniques are employed for structural vibration suppression, the overall controlled structure can be represented by Eq. (3-1)

$$M\ddot{q} + Kq = - C_C\dot{q} - K_Cq \qquad (3\text{-}1)$$

For positive definite M, $(K+K_C)$ and C_C, asymptotic stability of system (3-1) has been rigorously proven (see Section 3.5.4). This stability result can be applied to structures with decoupled-controllers as well as other general controlled systems represented by Eq. (3-1).

However, due to the high dimensionality of the vector q, matrix C_C is seldom positive definite; instead it is often positive semi-definite. The new stability result given in Section 3.5.3 provides a necessary and sufficient condition for system (3-1) to have asymptotic stability when C_C is only positive semi-definite. This result is also general and can be applied to any system represented by Eq. (3-1).

New insights into the robustness of stability properties of system (3-1) lead to some interesting interpretations. There are four matrices in Eq. (3-1) that could change or could be in error. But the basic sufficient conditions for stability of Eq. (3-1) are that

(1) M and $(K+K_C)$ are positive definite.

(2) C_C is positive semi-definite.

Thus, for robustness, M is free to change since M remains positive definite, whereas $(K+K_C)$ is constrained and must remain positive definite. For example, when $K_C \equiv 0$ (i.e., no position feedback), K could not become positive semi-definite. Similarly, there may be changes in C_C or K_C, but C_C must stay positive semi-definite. However, K_C may become even negative definite in theory and the system still retains stability, provided $(K+K_C)$ remains positive definite.

Again, robustness of stability results are general and applicable to all systems represented by Eq. (3-1).

If the decoupled-controller approach is used for LSS vibration suppression, the damping of residual modes may have a finite upper bound, as indicated in the illustrative example, whereas damping of critical modes could be increased to infinity in theory.

3.5 Appendices

3.5.1 Stability by Liapunov's Second Method

In this appendix, the main stability theorem of Liapunov is given only to the extent necessary to facilitate the discussion of the decoupled controller design technique. The material in the following is taken from Reference [5] where detailed discussions of Liapunov methods can be found.

Definitions

Consider systems governed by the vector differential equation

$$\frac{dx}{dt} = f(x,u(t),t), \quad -\infty < t < +\infty \qquad (3\text{-}14)$$

where vector x is the state of the system (3-14) and vector u(t) is the control function of Eq. (3-14). If the system (3-14) is free (i.e., unforced), then $u(t) \equiv 0$ for all t and

$$\frac{dx}{dt} = f(x,t) \tag{3-15}$$

Assume that there exists a unique vector function $x(t;x_o,t_o)$, differentiable in t, such that for any fixed x_o, t_o

(a) $x(t_o;\, x_o,\, t_o) = x_o$

(b) $\frac{dx}{dt}(t;\, x_o,\, t_o) = f(x(t;\, x_o,\, t_o),t)$

in some interval $|t-t_o| \leqq a(t_o)$. The function is called a solution of Eq. (3-15).

A state x_e of a free dynamic system (3-15) is an <u>equilibrium state</u> if

$$f(x_e;\, t) = 0 \text{ for all } t$$

or, equivalently

$$x(t;\, x_e,\, 0) = x_e \text{ for all } t$$

An equilibrium state x_e of a free dynamic system is <u>stable</u> if for every real number $\varepsilon > 0$ there exists a real number $\delta(\varepsilon,\, t_o) > 0$ such that $||x_o - x_e|| \leqq \delta$ implies

$$||x(t;\, x_o,\, t_o) - x_e|| \leqq \varepsilon \text{ for all } t \geqq t_o$$

An equilibrium state x_e of a free dynamic system is <u>asymptotically stable</u> if:

(a) It is stable.

(b) Every motion starting sufficiently near x_e converges to x_e as $t \to \infty$. In other words, there is some real constant $r(t_o) > 0$ and to every real number $\mu > 0$ there corresponds a real number $T(\mu,\, x_o,\, t_o)$ such that $||x_o-x_e|| \leqq r(t_o)$ implies

$$||x(t;\, x_o,\, t_o) - x_e|| \leqq \mu \text{ for all } t \geqq t_o + T$$

Consider Eq. (3-15) where $f(0,t) = 0$; namely, there is an equilibrium at the origin.

Theorem 1. If there exists a scalar function $V(x,t)$, with continuous first partial derivatives, satisfying the following conditions:

(a) $V(x,t) > 0$ for all $x \neq 0$ and all t; $V(0,t) = 0$ for all t

(b) $\dot{V}(x,t) \leq 0$ for all $x \neq 0$ and all t; $\dot{V}(0,t) = 0$ for all t,

then the origin of the system (3-15) is stable.

Theorem 2. If there exists a scalar function $V(x,t)$ with continuous first partial derivatives, satisfying the following conditions:

(a) $V(x,t) > 0$ for all $x \neq 0$ and all t; $V(0,t) = 0$ for all t

(b) $\dot{V}(x,t) \leq 0$ for all $x \neq 0$ and all t; $\dot{V}(0,t) = 0$ for all t

(c) $\dot{V}(x(t; x_o, t_o),t)$ does not vanish identically in $t \geq t_o$ for any t_o and any $x_o \neq 0$,

then the origin of the system (3-15) is asymptotically stable.

3.5.2 The Generalized Inverse of a Matrix

Theorem 3. If A is an m×n matrix (m>n) of rank n, then the solution of the equation $Ax = b$, where x is an n×1 vector and b is an m×1 vector, that minimizes the sum of squares of residuals $S = r^T r$, where $r = b - Ax$, is given by

$$x = (A^T A)^{-1} A^T b$$

where superscript "T" denotes the transpose and superscript "-1" denotes the inverse.

Theorem 4. If A is an m×n matrix (m<n) of rank m, then the solution of equation $Ax = b$, where x is an n×1 vector and b is an m×1 vector, that minimizes $x^T x$ is given by

$$x = A^T (AA^T)^{-1} b$$

The proofs of Theorems 2 and 3 can be found in Reference 1.

3.5.3 A New Stability Theorem (Contributed by Dr. James E. Potter of CSDL)

Consider a finite-element model for a large space structure as follows:

$$M\ddot{q} + Kq = f$$

where q is an n×1 vector and M and K are mass and stiffness matrices respectively. Assume that the force f applied to the structure can be expressed mathematically as

$$f = -C_C\dot{q}$$

where C_C is an n×n symmetric positive semi-definite matrix, and that M and K are both symmetric and positive definite.

<u>Theorem</u>. Denote $P \triangleq M^{-1}K$. Then the following system

$$M\ddot{q} + C_C\dot{q} + Kq = 0 \tag{3-16}$$

where $M > 0$, $K > 0$, and $C_C \geq 0$, is asymptotically stable if and only if (P, C_C) is an observable pair, i.e., if and only if the matrix

$$Q^T \triangleq [C_C^T \vdots (C_CP^2)^T \vdots \ldots \vdots (C_CP^{n-1})^T]$$

has rank n.

<u>Proof</u>.

(1) To prove that the observability pair (P, C_C) implies the asymptotic stability of system (3-16), suppose (3-16) is not asymptotically stable. Then there exist solutions to system (3-16) in the following form

$$q = b \cos \omega t, \; b \neq 0 \tag{3-17}$$

where b is an n×1 real constant vector, since

$$L \triangleq \dot{q}^TM\dot{q} + q^TKq$$

is a Liapunov function for system (3-16) and (3-16) must be stable. Substituting Eq. (3-17) into (3-16), the following equation results:

$$(-\omega^2M+K)b \cdot \cos \omega t - \omega \cdot C_C \cdot b \cdot \sin \omega t = 0, \; t \geq 0 \tag{3-18}$$

If $\omega = 0$, then equation (3-18) implies that

$$Kb = 0$$

which contradicts the assumption that K is positive definite. Thus $\omega \neq 0$. With $\omega \neq 0$ and the orthogonality of $\sin \omega t$ and $\cos \omega t$, Eq. (3-18) implies that

$$\begin{cases} (-\omega^2 M+K)b = 0 & (3\text{-}19) \\ C_C b = 0 & (3\text{-}20) \end{cases}$$

and

$$b = \frac{1}{\omega^2} Pb \qquad (3\text{-}21)$$

Substituting equations (3-21) into equation (3-20) repeatedly, we have

$$C_C P^i b = 0, \; i = 1, 2, \ldots$$

Therefore

$$Qb = 0.$$

However, Q is an $n^2 \times n$ matrix and thus must have rank less than n since it has a right null vector b. Thus (P, C_C) is not observable.

(2) To prove that asymptotic stability of (3-16) implies observability of (P, C_C), suppose that (P, C_C) is not observable. Then there exists a nonzero vector b such that

$$Qb = 0$$

that is

$$C_C P^i b = 0 \text{ for } i = 0, 1, \ldots, n-1 \qquad (3\text{-}22)$$

By the Cayley-Hamilton theorem and Eq. (3-18), we also have

$$C_C P^i b = 0 \text{ for } i = n, n+1, \ldots \qquad (3\text{-}23)$$

Recall that M and K are symmetric and positive definite and that $P = M^{-1}K$. Therefore there exists a transformation Φ such that

$$\Phi^T M \Phi = I$$

$$\Phi^T K \Phi = \text{diag}\,(\lambda_1, \ldots, \lambda_n), \quad \lambda_1, \ldots, \lambda_n > 0$$

$$\equiv \text{diag}\,(\lambda_i), \quad i = 1, \ldots, n$$

Thus

$$M = \Phi^{-T}\Phi^{-1}$$

$$K = \Phi^{-T}\,\mathrm{diag}(\lambda_i)\Phi^{-1}$$

Therefore

$$P \triangleq M^{-1}K = \Phi\Phi^T\Phi^{-T}\,\mathrm{diag}(\lambda_i)\Phi^{-1} = \Phi\,\mathrm{diag}\,(\lambda_i)\Phi^{-1} \tag{3-24}$$

Now, define an (n-1)-th degree polynomial $\alpha(x)$ as follows

$$\alpha(x) \triangleq \sum_{k=1}^{n} \frac{\sqrt{\lambda_k}\,\alpha_k(x)}{\alpha_k\,(\lambda_k)}$$

where

$$\alpha_k(x) \triangleq \prod_{\substack{j=1\\ j\neq k}}^{n} (x-\lambda_j)$$

Then

$$\alpha(\lambda_i) = \sqrt{\lambda_i},\ i = 1,\dots,n \tag{3-25}$$

Define

$$G \triangleq \Phi\,\mathrm{diag}(\sqrt{\lambda_i})\Phi^{-1}$$

From Eq. (3-25), G can be expressed in the following form

$$\begin{aligned} G &= \Phi\{\mathrm{diag}\,[\alpha(\lambda_i)]\}\Phi^{-1} \\ &= \Phi\{\alpha[\mathrm{diag}(\lambda_i)]\}\Phi^{-1} \\ &= \alpha[\Phi\,\mathrm{diag}(\lambda_i)\Phi^{-1}] = \alpha(P) \end{aligned} \tag{3-26}$$

where the last equality is obtained from equation (3-24). From the definition of $\alpha(x)$ and equation (3-26), there are coefficients $a_o,\dots, a_{n-1}$ such that

$$G = \sum_{i=o}^{n-1} a_i \cdot P^i \tag{3-27a}$$

and

$$G^2 = P \tag{3-27b}$$

Let

$$\tilde{q} \triangleq \cos(Gt) \cdot b \tag{3-28}$$

then

$$\frac{d^2\tilde{q}}{dt^2} = -G^2\tilde{q} = -P\tilde{q} = -M^{-1}K\tilde{q}$$

or, equivalently,

$$M\ddot{\tilde{q}} + K\tilde{q} = 0 \tag{3-29}$$

Furthermore, Eqs. (3-27) and (3-23) imply that

$$C_C G^{2i} b = C_C P^i b = 0, \; i = 0, 1, 2, \ldots$$

Therefore

$$C_C \dot{\tilde{q}} = - C_C G \sin(Gt) \cdot b = 0$$

Combining this equation with equation (3-25) yields

$$M\ddot{\tilde{q}} + C_C\dot{\tilde{q}} + K\tilde{q} = 0 \tag{3-30}$$

But the solution to this equation, which is given in Eq. (3-28) does not approach 0 as time t approaches infinity. Thus the system (3-30), which is identical to Eq. (3-16), is not asymptotically stable.

3.5.4 Stability and Robustness of the Controlled Structure

3.5.4.1 Stability

Consider the closed-loop system equation of the following form

$$M\ddot{q} + Kq = -C_C\dot{q} - K_Cq \tag{3-31}$$

which is equivalent to

$$M\ddot{q} + C_C\dot{q} + (K+K_C)q \quad = \quad 0 \tag{3-32}$$

Sufficient conditions for stability of the closed-loop system can be established by applying the Liapunov direct method to Eq. (3-32); stability conditions can be stated as follows:

<u>Theorem</u>. Consider Eq. (3-32) where M and $(K+K_C)$ are symmetric and positive definite. Then, if C_C is positive semi-definite, the system (3-32) is stable in the sense of Liapunov. If C_C is positive definite, the system 3-32 is asymptotically stable.

<u>Proof</u>. Consider a Liapunov testing function (L) with the following form

$$L \quad = \quad \frac{1}{2}\dot{q}^T M\dot{q} + \frac{1}{2}q^T(K+K_C)q \tag{3-33}$$

which is positive for $\dot{q}\neq 0$ and $q\neq 0$.

Then the rate of change of the Liapunov testing function is

$$\begin{aligned} \dot{L} &= \dot{q}^T M\ddot{q} + \dot{q}^T(K+K_C)q \\ &= \dot{q}^T[M\ddot{q} + (K+K_C)q] \end{aligned} \tag{3-34}$$

From Eq. (3-32), Eq. (3-34) can be simplified as

$$\dot{L} \quad = \quad -\dot{q}^T C_C\dot{q} \tag{3-35}$$

Equations (3-33) and (3-35) show that L is indeed a Liapunov function [5] for system (3-32), if C_C is positive semi-definite. Therefore, from Theorem 1 in Section 3.5.5, stability of system (3-32) follows.

To prove the asymptotic stability result with Eqs. (3-33) and (3-35) when C_C is positive definite, Theorem 2 in Section 3.5.1 will be used.

First from Eqs. (3-33) and (3-35), it is apparent that

(1) $L(q,\dot{q}) > 0$ for all $q \neq 0$ and $\dot{q} \neq 0$

(2) $\dot{L}(q,\dot{q}) = -\dot{q}^T C_C\dot{q} \leq 0$ for all q and $\dot{q}$.

To verify that $\dot{L}(q,\dot{q}) = \dot{L}(\dot{q})$ does not vanish identically in $t \geq t_o$ for any t_o and any $q(t_o) \neq 0$ and $\dot{q}(t_o) \neq 0$, a proof by contradiction is given. Assume that for some t_o and some $q(t_o) \neq 0$ and $\dot{q}(t_o) \neq 0$:

$$\dot{L}(q,\dot{q}) \quad = \quad \dot{L}(\dot{q}) \quad \equiv \quad 0 \text{ for } t \geq t_o \tag{3-36}$$

Then Eq. (3-35) implies that

$$\dot{q} = 0 \text{ for } t \geq t_o \text{ since } C_C \text{ is positive definite} \tag{3-37}$$

Equation (3-37), however, implies that

$$\ddot{q} = 0 \text{ for } t > t_o$$

From the system Eq. (3-32) then,

$$(K+K_C)q = 0 \text{ for } t > t_o$$

However, since $(K+K_C)$ is also positive definite, it is concluded that

$$q \equiv 0 \text{ for } t > t_o \tag{3-38}$$

which together with (3-37) implies $q(t_o) = \dot{q}(t_o) = 0$ contradicting the assumption that $q(t_o) \neq 0$ and $\dot{q}(t_o) \neq 0$. Consequently the conditions of Theorem 2 in Section 3.5.1 are satisfied and asymptotic stability is therefore assured.

3.5.4.2 Robustness

Given a constant gain feedback control law as specified in Eq. (3-31), it is interesting to investigate the requirement of the control law to have stability robustness against model parameter errors. Consider the following system

$$\hat{M}\ddot{q} + \hat{K}q = f \tag{3-39}$$

where $\hat{M}$ represents a new mass matrix and $\hat{K}$ a new stiffness matrix. However, f remains the same as given in Eq. (3-31).

To establish sufficient stability criteria, the direct method of Liapunov can again be applied to Eq. (3-39) with a new Liapunov testing function L defined as

$$\hat{L} = \frac{1}{2}\dot{q}^T\hat{M}\dot{q} + \frac{1}{2}q^T(\hat{K}+K_C)q$$

which is positive for $\dot{q} \neq 0$ and $q \neq 0$, if $\hat{M}$ and $(\hat{K}+K_C)$ are positive definite. The time derivative of $\hat{L}$ along trajectories governed by Eq. (3-39) is

$$\dot{\hat{L}} = -\dot{q}^T C_C \dot{q}$$

if $(\hat{K}+K_C)$ and $\hat{M}$ are symmetric.

In general, $\hat{M}$ is symmetric and positive definite whereas $\hat{K}$ is symmetric and positive semi-definite. Therefore, robustness of stability against parameter errors can be assured if controller gains are such that

(a) $(\hat{K}+K_C)$ is symmetric and positive definite.

(b) C_C is positive definite or positive semi-definite.

It should be noted that these are only sufficient conditions for robustness. Furthermore, for a general position feedback gain matrix K_C, condition (a) above constrains the degree of tolerable parameter errors in $\hat{K}$ and therefore no absolute robustness can be stated. However, if K_C is symmetric and positive definite, then absolute robustness of the controlled system is obtained such that the system remains stable regardless of what parameter errors might be in the mass and stiffness matrices. For systems without rigid-body modes (i.e., the stiffness matrix is symmetric and positive definite), absolute robustness can also be obtained if K_C is symmetric and positive semi-definite. This result is applicable to the case when only velocity feedback (i.e., $K_C = 0$) is employed for the control of systems without rigid-body modes.

References

1. Noble, B., Applied Linear Algebra, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1969.

2. LaSalle, J., and S. Lefschetz, Stability by Liapunov's Direct Method with Applications, Academic Press, Inc., New York, New York, 1961.

3. Canavin, J.R., "The Control of Spacecraft Vibrations Using Multivariable Output Feedback," Report R-1137, The Charles Stark Draper Laboratory, Inc., Cambridge, Ma., January 1978; Also presented at AIAA Guidance and Control Conference, Palo Alto, Ca., August 1978.

4. Quartararo, R., "Modal Control of Flexible Spacecraft," Adv. Tech. Lab. Prog. for Large Space Structures, I, App. B, Rockwell International SD-76-SA-0210, November 1976.

5. Ogata, Katsuhiko, State Space Analysis of Control Systems, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1967.

SECTION 4

OUTPUT FEEDBACK CONTROL VIA DAVISON-WANG METHOD OF POLE ASSIGNMENT

4.1 Background

4.1.1 Brief Introduction of the Main Ideas

The objective of the Davison-Wang method is to find a constant output-feedback gain matrix G for system (2-12) - (2-13) that assigns $p \triangleq \min \{n, m + \ell - 1\}$ closed-loop poles to p desired locations in the complex plane. In algebraic terms, the objective is to find a matrix G such that the closed-loop system matrix of (2-12) - (2-13) has p desired eigenvalues. Nothing can be said about the remaining n-p poles.

If $\max \{m, \ell\} = \min \{n, m + \ell - 1\}$, then p desired closed-loop poles are assigned in one step using Davison's construction algorithm. On the other hand, if $\max \{m, \ell\} < \min \{n, m + \ell - 1\}$, then p desired closed-loop poles are assigned in three steps: first assign $p_1 \triangleq \max \{m, \ell\}$ desired closed-loop poles using Davison's construction algorithm, "seal" all but one of those assigned by making them unobservable (or uncontrollable), and then assign the remaining $p_2 \triangleq \min \{n, m + \ell - 1\} - p_1 + 1$ desired closed-loop poles again using Davison's construction algorithm.

Note that when $m + \ell \geq n + 1$, the Davison-Wang method can assign all n closed-loop poles of the system.

Davison's construction algorithm is formulated for multiple-input multiple-output systems, but makes explicit use of the simplicity in single-input systems. In a single-input system, the coefficients of the characteristic equation are <u>linear</u> functions of the feedback gains used for amplifying and combining the outputs. Thus, the feedback gains required for implementing the assignment of desired closed-loop poles can be found by solving a set of simultaneous <u>linear</u> algebraic equations. A multiple-input system will lose such simplicity (i.e., the linearity) unless it is simplified to a single-input system. Davison's construction algorithm starts by converting a multiple-input system to a single-input system.

For a system with $\ell \geq m$ (i.e., more outputs than inputs), m inputs are reduced to one input, and ℓ outputs are fed back through the single input. Then as many as ℓ desired closed-loop poles can be assigned by the Davison construction algorithm. On the other hand, for a system with $m \geq \ell$ (i.e., more inputs than outputs), a dual approach is used. Namely, ℓ outputs are reduced to one output and fed back through the m inputs. As many as m desired closed-loop poles are then assignable by Davison's construction algorithm. To summarize, the $m \times \ell$ gain matrix G in either case is considered to be a dyadic product of two vectors, $G = \theta g^T$ where θ is an m-vector, g is an ℓ-vector, and superscript "T" denotes transpose.

Distinct eigenvalues of a diagonal (or diagonalized) system matrix are made unobservable from the outputs fed back by making the corresponding columns of the observation matrix zero. Desired system poles can thereby be

frozen, i.e., protected from unwanted alteration. Output feedback with any arbitrary full matrix of constant gains is introduced to make the resultant closed-loop system have distinct eigenvalues.

4.1.2 Brief Introduction of the Underlying Theory

4.1.2.1 Davison's Theorem

If the system (2-12) - (2-13) is completely controllable, if the eigenvalues of the system matrix A are distinct, or are repeated but no two Jordan blocks correspond to a common eigenvalue, and if the observation matrix C has rank $\ell \leq n$, then Davison [1] showed that a feedback of the outputs in the form

$$u = Gy \tag{4-1}$$

where G is a constant gain matrix can always be found so that ℓ eigenvalues of the closed-loop system matrix A+BGC are arbitrarily close (but not necessarily equal) to ℓ pre-assigned (or desired) complex-conjugate values.

An algorithm for constructing the feedback gain matrix G was proposed.

4.1.2.2 Davison and Chatterjee's Theorem

The purpose of the above assumption on the eigenvalues of the system matrix A was to make a system which is completely controllable from multiple inputs also completely controllable from a single input. Using a theorem of Brasch and Pearson [5], Davison and Chatterjee [2] then modified Davison's original theorem as follows. If the system is both completely controllable and completely observable, and if the control matrix B has rank $m \leq n$ and the observation matrix C has rank $\ell \leq n$, then a linear constant-gain feedback of the outputs in the form (4-1) can always be found so that max $\{m, \ell\}$ eigenvalues of the closed-loop system matrix A+BGC are arbitrarily close (but not necessarily equal) to max $\{m, \ell\}$ pre-assigned complex-conjugate values.

The theorem of Brasch and Pearson says that if a system is both completely controllable and completely observable, then there exists a constant gain matrix K such that the closed-loop system is both completely controllable from a single input and completely observable from a single output. So, introducing an additional output feedback will make a system completely single-input controllable and completely single-output observable, if it is initially not so.

4.1.2.3 Davison and Wang's Theorem on Simple Poles

If a system has no multiple poles (i.e., if its system matrix has no repeated eigenvalues), then whenever it is completely (multiple-input) controllable it is also completely single-input controllable, and similarly, whenever it is completely (multiple-output) observable it is also completely single-output observable. Davison and Wang [6] showed that if a system is both completely controllable and completely observable, then almost any constant-gain output feedback will make the closed-loop system matrix have distinct

eigenvalues. Davison and Chow [3] used these facts in simplifying Davison's original construction algorithm.

4.1.2.4 Davison and Wang's Theorem on Pole Assignment

Davison's theorem was extended by Davison and Wang [4] as follows. Given any system satisfying the assumptions in Davison and Chatterjee's theorem (see Section 4.1.2.1, or [2]), there exists, for almost all (B,C) pairs, a constant gain feedback of the outputs in the form Eq. (4-1) such that the closed-loop system matrix A+BGC has min $\{n, m + \ell-1\}$ eigenvalues assigned arbitrarily close to min $\{n, m + \ell-1\}$ specified complex-conjugate values. This implies that almost all linear time-invariant multivariable systems can be stabilized by using only output feedback with constant gains, provided that $m + \ell \geq n + 1$.

An algorithm, similar to what Topaloglu and Seborg [7] proposed earlier, was also given in [4] for finding the gain matrix for assigning min $\{n, m + \ell-1\}$ closed-loop poles. For the case $\ell \geq m$ and $m + \ell \leq n + 1$, Topaloglu and Seborg's algorithm involves three steps: assignment of ℓ poles, "protection" of m-1 poles by making them uncontrollable, and assignment of ℓ additional poles. The method for assigning the ℓ poles is the same for both the first and the last steps, and is essentially the same as Davison's construction algorithm. For the same case, the Davison and Wang algorithm also involves three steps: assignment of ℓ poles, "protection" of ℓ-1 poles by making them unobservable, and assignment of m poles. The difference lies in the second and the third steps.

4.1.3 Outline of the Design Method

4.1.3.1 Davison's Construction Algorithm for the Assignment of max $\{m,\ell\}$ Poles

For notational convenience, assume $\ell \geq m$ (more outputs than inputs), so that $\ell = \max\{m, \ell\}$. Assume also that the eigenvalues of the system matrix A are distinct; otherwise, introduce an arbitrary constant-gain output feedback to make them all distinct (see Section 4.1.2.3 above).

4.1.3.1.1 Consider Single Inputs - Suppose that the m inputs $u_1, \ldots, u_m$ are generated by a single input v, and the ℓ outputs $y_1, \ldots, y_\ell$ are simplified and combined to produce the single input v. That is

$$u = \theta v, \; v = g^T y \tag{4-2}$$

where $\theta = (\theta_1, \ldots, \theta_m)$ is an m-vector (of amplifier gains), and $g = (g_1, \ldots, g_\ell)$ is an ℓ-vector (of amplifier gains). Both θ and g are to be determined. Then, from (4-1) and (4-2), the $m \times \ell$ matrix of output feedback gains is given by the dyadic product of vectors θ and g, namely

$$G = \theta g^T \tag{4-3}$$

Therefore, the construction of the gain matrix G for desired pole assignment reduces to the construction of an m-vector θ and an ℓ-vector g.

4.1.3.1.2 Construct an m-vector θ - Using a similarity transformation T and input (4-2), the state Eq. (2-12) becomes

$$\dot{z} = \tilde{A}z + \tilde{B}\theta v \tag{4-4}$$

where

$$\tilde{A} = T^{-1}AT, \text{ a diagonal matrix}$$

$$\tilde{B} = T^{-1}B$$

Determine numbers $\theta_1, \ldots, \theta_m$ such that each of the n components of the n-vector $\xi \triangleq \tilde{B}\theta$ is nonzero. With any such m-vector θ, the state equation (2-12) becomes

$$\dot{x} = Ax + bv$$

where (4-5)

$$b = B\theta$$

Such a single-input system is completely controllable from the single input v.

4.1.3.1.3 Transform Matrix A to Companion Form - For constructing an ℓ-vector g for pole assignment, it is convenient to transform matrix A to companion form

$$\hat{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & & \cdot \\ \cdot & & \ddots & \cdot \\ \cdot & & & \cdot \\ \cdot & & \cdot & 1 \\ a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

where $a_1, \ldots, a_n$ are the coefficients of the characteristic equation

$$\lambda^n = a_1 + a_2\lambda + \cdots + a_n\lambda^{n-1} \tag{4-6}$$

These coefficients are to be determined as follows. By the Cayley-Hamilton Theorem

$$A^n = a_1 I + a_2 A + \cdots + a_n A^{n-1}$$

Post-multiplying $b \triangleq B\theta$ yields

$$A^n b = a_1 b + a_2 Ab + \cdots + a_n A^{n-1} b = Q \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

where

$$Q = [b, Ab, \dots, A^{n-1}b]. \tag{4-7}$$

The matrix Q is the controllability matrix of the single-input system (4-5). Since it is invertible, we thus get

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = Q^{-1}A^n b \tag{4-8}$$

The matrix for transforming A to $\hat{A}$ is given by the matrix product $Q\hat{Q}$, where

$$\hat{Q} = \begin{bmatrix} -a_2 & -a_3 & \cdots & -a_n & 1 \\ -a_3 & -a_4 & \cdots & \cdot^{\cdot^{\cdot}} & \vdots \\ -a_4 & \vdots & \cdot^{\cdot^{\cdot}} & & \vdots \\ \vdots & \vdots & & & \vdots \\ -a_n & 1 & & & \vdots \\ 1 & 0 & \cdots & & 0 \end{bmatrix} \tag{4-9}$$

Let $\hat{b} \triangleq (Q\hat{Q})^{-1}b \equiv \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$, $\quad \hat{C} \triangleq CQ\hat{Q}$

Then by setting $x = Q\hat{Q}z$, with $z_1, \dots, z_n$ being the new coordinates, the single-input system (4-5) becomes

$$\begin{aligned} \dot{z} &= \hat{A}z + \hat{b}v \\ y &= \hat{C}z \end{aligned} \tag{4-10}$$

4.1.3.1.4 Derive The Closed-Loop Characteristic Equation - With the output feedback (4-2), the system matrix of the closed-loop system (4-10) is given by

$$\hat{A} + \hat{b}g^T\hat{C} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ a_1 + \delta_1 & a_2 + \delta_2 & \cdots & & a_n + \delta_n \end{bmatrix} \tag{4-11}$$

where $[\delta_1, \dots, \delta_n] = g^T CQ\hat{Q}$.

Since it is also in the companion form, its characteristic equation is similarly given by

$$\lambda^n = (a_1 + \delta_1) + (a_2 + \delta_2)\lambda^1 + \cdots + (a_n + \delta_n)\lambda^{n-1} \tag{4-12}$$

Rewriting it we have

$$\begin{aligned} &\lambda^n - a_n\lambda^{n-1} - \cdots - a_2\lambda - a_1 \\ &= \delta_1 + \delta_2\lambda + \cdots + \delta_n\lambda^{n-1} \\ &= [\delta_1, \delta_2, \ldots, \delta_n] \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-1} \end{bmatrix} = g^T C Q\hat{Q} \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-1} \end{bmatrix} \end{aligned}$$

Consequently

$$\Lambda(\lambda) = g^T C Q\hat{Q} h(\lambda) \tag{4-13}$$

where

$$\Lambda(\lambda) = \lambda^n - a_n\lambda^{n-1} \cdots - a_2\lambda - a_1$$
$$h(\lambda) = (1, \lambda, \ldots, \lambda^{n-1})$$

4.1.3.1.5 Construct an ℓ-Vector for Assignment of Desired Closed-Loop Poles

Let $\lambda_1, \ldots, \lambda_\ell$ denote the ℓ desired closed-loop poles. The ℓ-vector of gains required for implementing the pole assignment is to be found from (4-13) by substituting in these eigenvalues.

Substituting λ_i in (4-13) yields

$$\Lambda_i = g^T C Q\hat{Q} h_i \qquad i = 1, \ldots, \ell \tag{4-14}$$

where

$$\Lambda_i \triangleq \Lambda(\lambda_i) = \lambda_i^n - a_n\lambda_i^{n-1} - \cdots - a_2\lambda_i - a_1 \tag{4-15}$$

$$h_i \triangleq h(\lambda_i) = (1, \lambda_i, \lambda_i^2, \ldots, \lambda_i^{n-1}) \tag{4-16}$$

If λ_j is the r^{th} repeat of λ_i, then differentiating (4-13) r times and substituting in λ_i gives

$$\frac{d^r}{d\lambda^r} \Lambda(\lambda_i) = g^T CQ\hat{Q} \frac{d^r}{d\lambda^r} h(\lambda_i).$$

Thus, the number Λ_j and the vector h_j in (4-14) should be redefined as

$$\Lambda_j = \frac{d^r}{d\lambda^r} \Lambda(\lambda_i) \tag{4-15'}$$

$$h_j = \frac{d^r}{d\lambda^r} h(\lambda_i) \tag{4-16'}$$

to increase the number of independent equations in (4-14). Consequently, we have

$$\Lambda = g^T CQ\hat{Q}H \tag{4-17}$$

where

$\Lambda = [\Lambda_1, \ldots, \Lambda_\ell]$, a row vector of ℓ components

$H = [h_1, \ldots, h_\ell]$, an $n \times \ell$ matrix of ℓ n-vectors.

Therefore, the desired vector g of feedback gains can be solved from (4-17) as

$$g^T = \Lambda(CQ\hat{Q}H)^{-1} \tag{4-18}$$

provided that the matrix $S = CQ\hat{Q}H$ is invertible.

4.1.3.1.6 Summary of the Algorithm for the Case $\ell \geq m$

Step 1: Compute the eigenvalues and eigenvectors of matrix A.

Step 2: Define the orthogonal transformation T by the normalized eigenvectors. Compute the inverse T^{-1}.

Step 3: Compute matrix $\tilde{B} = T^{-1}B$.

Step 4: Determine an m-vector θ so that each component of vector $\tilde{B}\theta$ is nonzero.

Step 5: Compute vector $b = B\theta$.

Step 6: Compute matrix Q defined by (4-7), and the inverse Q^{-1}.

Step 7: Compute coefficients $a_1,\ldots,a_n$ using (4-8).

Step 8: Compute matrix $C Q\hat{Q}$ with $\hat{Q}$ defined by (4-9).

Step 9: Compute numbers Λ_i and vectors h_i by substituting desired closed-loop poles $\lambda_1,\ldots,\lambda_\ell$ into (4-15) - (4-16), and (4-15') - (4-16') for repeated values. Form the row vector Λ and the matrix H.

Step 10: Compute the matrix $CQ\hat{Q}H$ and its inverse $(CQ\hat{Q}H)^{-1}$.

Step 11: Compute the row vector $g^T = \Lambda(CQ\hat{Q}H)^{-1}$.

Step 12: Compute the gain matrix $G = \theta g^T$.

4.1.3.1.7 Extension to the Case $\ell \leq m$. To assign $m = \max\{m,\ell\}$ closed-loop poles for this case, the preceeding algorithm (for the case $\ell \geq m$) can be applied directly if the duality between control and observation is employed. Specifically, (1) replace matrix B in the algorithm by the transpose C^T, matrix C by the transpose B^T, and matrix A by its transpose A^T. Consequently, the numbers ℓ and m should also be interchanged. (2) Follow the algorithm and obtain an $\ell \times m$ gain matrix G_d required for assignment of m desired closed-loop poles to the dual system

$$\begin{cases} -\dot{x}_d = A^T x_d + C^T u_d \\ y_d = B^T x_d \end{cases}$$

by "output feedback"

$$u_d = G_d y_d$$

(3) The desired output-feedback gain matrix for the original system is given by

$$G = G_d^T$$

4.1.3.2 Davison-Wang Algorithm for the Assignment of $\min\{n, m + \ell - 1\}$ Poles

Assume that $\max\{m,\ell\}$ desired closed-loop poles have been assigned using Davison's construction algorithm. Let G_1 denote the required gain matrix. If $\max\{m,\ell\} = \min\{n,m + \ell-1\}$, then stop, since no more poles can be assigned.

Consider the case max $\{m,\ell\} < \min \{n,m + \ell-1\}$. To assign more desired closed-loop poles while retaining the assigned and the desired, it is necessary to "seal" the latter. Assume that $\ell \geq m$ again for notational convenience, and that $\ell \geq 2$ for nontriviality. Assume that all the eigenvalues are distinct; otherwise, add any output feedback to make them distinct. Apply a coordinate transformation T such that the system matrix $A+BG_1C$ is diagonalized

$$T^{-1}(A+BG_1C)T = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}$$

where $D_1 = \text{diag}\ \{\lambda_1,\ldots,\lambda_t\}$ and $D_2 = \text{diag}\ \{\lambda_{t+1},\ldots,\lambda_n\}$. In terms of the new coordinates $\xi_1,\ldots,\xi_n$, the system becomes

$$\dot{\xi} = \left[\begin{array}{c|c} D_1 & 0 \\ \hline 0 & D_2 \end{array}\right]\xi + \left[\begin{array}{c} \overline{B}_1 \\ \hline \overline{B}_2 \end{array}\right]u \tag{4-19}$$

$$y = [\overline{C}_1 \vdots \overline{C}_2]\ \xi$$

$$x = T\xi$$

where $\overline{B}_1$ and $\overline{B}_2$ are of dimension $t \times m$ and $(n-t) \times m$, respectively, such that

$$\begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix} = T^{-1}B$$

$\overline{C}_1$ and $\overline{C}_2$ are of dimension $\ell \times t$ and $\ell \times (n-t)$, respectively, such that $[\overline{C}_1 \vdots \overline{C}_2] = CT$.

Assume that

$$\text{rank}\ (\overline{B}_2) = \min\ \{m,n-t\}, \text{ and} \tag{4-20}$$

$$\text{rank}\ (\overline{C}_1,d_j) = \text{rank}\ (\overline{C}_1) + 1,\ j = 1,\ldots,n-t \tag{4-21}$$

where d_j denotes the j^{th} column of matrix $\overline{C}_2$. Then choose an ℓ-vector θ such that

$$\theta^T\overline{C}_1 = 0 \tag{4-22}$$

and

$$\theta^T d_j \neq 0,\ j = 1,\ldots,n-t \tag{4-23}$$

System (4-19) is thus converted to a single-output system as follows

$$\dot{\xi} = \left[\begin{array}{c|c} D_1 & 0 \\ \hline 0 & D_2 \end{array}\right] \xi + \left[\begin{array}{c} \bar{B}_1 \\ \hline \bar{B}_2 \end{array}\right] u$$

$$\tilde{y} = [0 \mid \tilde{d}]\xi$$

where $\tilde{y} \triangleq \theta^T y$ is a scalar, and $\tilde{d} = \theta^T \bar{C}_2$ is a row vector of nonzero components. It is not difficult to see that D_1 is the unobservable part of the system. Hence eigenvalues $\lambda_1, \ldots, \lambda_t$ cannot be altered by feedback control using output $\tilde{y}$. On the other hand, by assumption (4-20) and condition (4-23), the subsystem

$$\begin{cases} \dot{\xi}^2 = D_2 \xi^2 + \bar{B}_2 u \\ \tilde{y} = \tilde{d}\xi_2 \end{cases} \tag{4-24}$$

is both completely controllable and completely observable. Notice that this is the case where there are more inputs than outputs. Apply Davison's construction algorithm for this case (see Section 4.1.3.1.7) to subsystem (4-24) and get the output feedback

$$u = g\tilde{y} \triangleq g\theta^T y \tag{4-25}$$

for assigning min {m,n-t} desired closed-loop poles to subsystem (4-24). Equivalently, this feedback control assigns these closed-loop poles to the completely observable part, namely D_2, of system (4-19). The required gain matrix is thus given as

$$G_2 = g\theta^T$$

To summarize, the sum of the two gain matrices,

$$G = G_1 + G_2$$

is the required feedback gain matrix for assigning totally t + min {m,n-t} $\equiv$ min {m + t, n} $\equiv$ min {m + ℓ-1, n} desired closed-loop poles to system (2-12) - (2-13).

In case m $\geq$ ℓ and m $\geq$ 2, apply the above procedure to the dual system first. Then the transpose of the gain matrix obtained is the desired matrix for this case.

4.1.4 Summary of Assumptions Made

(1) The system (2-12) - (2-13) is completely controllable and completely observable.

(2) The matrix B has rank $m \leq n$.

(3) The matrix C has rank $\ell \leq n$.

(4) The set of desired closed-loop poles to be assigned is complex conjugate: any complex numbers appear in complex conjugate pairs.

(5) The $(n-\ell + 1) \times m$ matrix $\overline{B}_2$ has full rank - see assumption (4-20).

(6) No column of the $\ell \times (n-\ell + 1)$ matrix $\overline{C}_2$ is a linear combination of the columns of the $\ell \times (\ell-1)$ matrix $\overline{C}_1$ - see assumption (4-21).

4.1.5 Summary of Technical Tricks Used

(1) Conversion of a multiple-input multiple-output system to either a single-input multiple-output system or a multiple-input single-output system.

(2) Restriction of the $m \times \ell$ feedback gain matrix G to be a dyadic product of an m-vector θ and an ℓ-vector g: $G = \theta g^T$.

(3) "Saving" of desired system poles by making them unobservable from some scalarized output.

4.2 Discussion

4.2.1 Strengths

(1) By definition, the objective of modal control is to control certain modes of the system response by altering them with feedback. Decay rates and vibration frequencies of the system response are determined by the location of the system poles. Output feedback control by the Davison-Wang method enables the designer to assign a set of desired closed-loop poles and hence to have direct control over the modes of the system response.

(2) The method is an extension of classical frequency-domain design techniques to multivariable systems. However, the approach is modern, analytical, and systematic; the concept is simple and the algorithm is straightforward.

(3) The dyadic form of the output-feedback gain matrix G is simpler to implement than the general form. Figure 4-1 shows a typical Davison output-feedback controller, whose gain matrix is in dyadic form, whereas Figure 4-2 shows a general form of output-feedback controller. The simplicity of the former is evident.

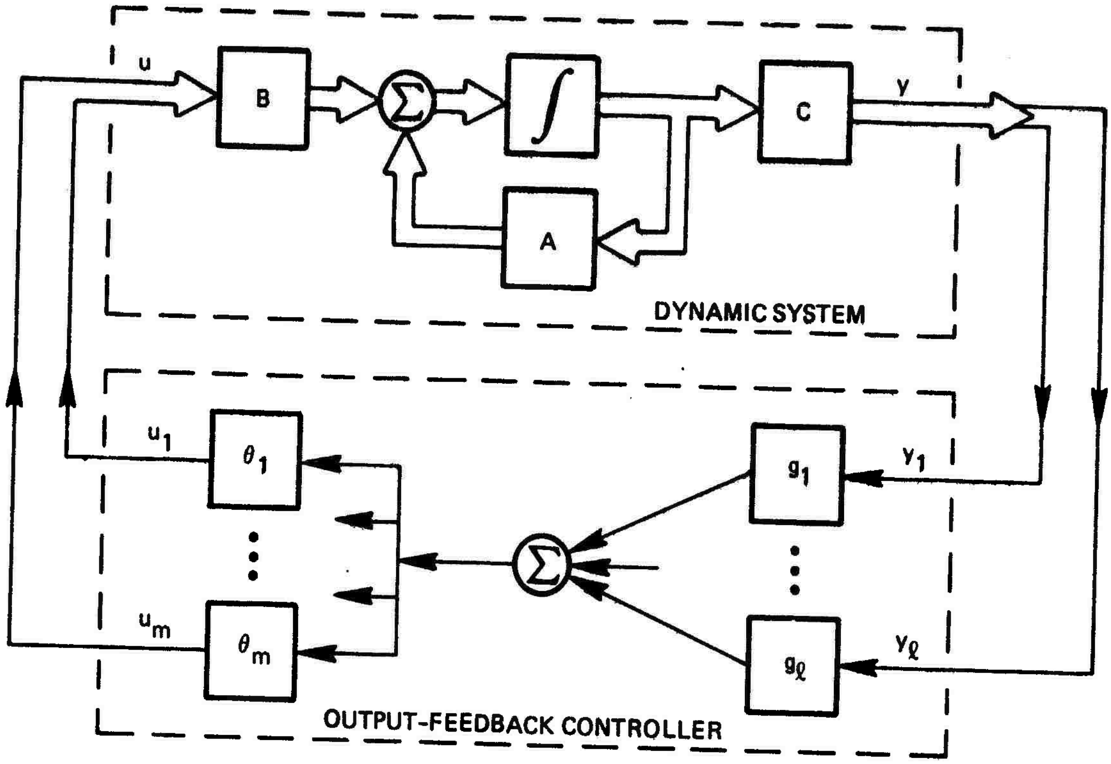


Figure 4-1. Dyadic form of output feedback control, $u = \theta g^T y$.

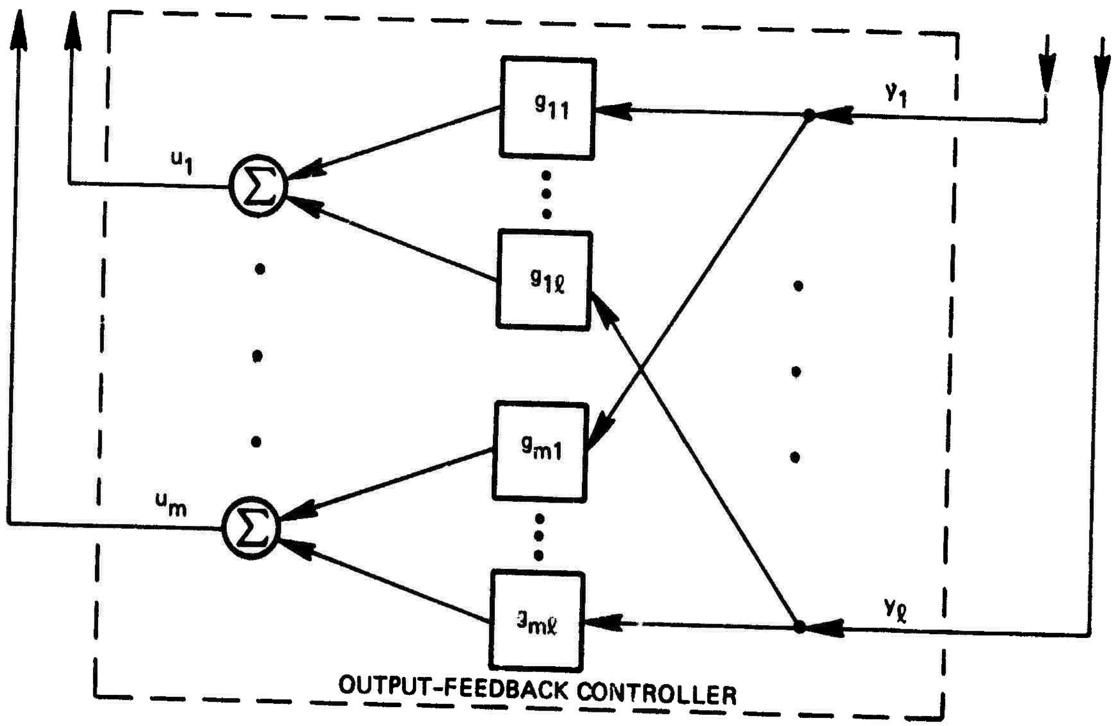


Figure 4-2. General form of output feedback control, $u = Gy$.

(4) The size of the computer memory and the amount of processing time required in the design process are much lower than the corresponding requirements for solving $n \times n$ matrix Riccati equations in the design of linear-quadratic regulators and observers.

(5) Like other output-feedback controllers, Davison-Wang output-feedback controllers are implementable by electronic or electro-mechanical hardware. Moreover, no dynamic state estimators are required in the feedback loops and the associated disadvantages are avoided. Note that a dynamic state estimator requires a (hardware or software) simulation of the system model. The addition of a dynamic state estimator increases the sensitivity of the closed-loop system to model errors and the order of the closed-loop system. Moreover, the reliance upon on-line computation is increased if the estimator is implemented by computer software.

4.2.2 Weaknesses

4.2.2.1 Theoretical Limitations

(1) At most min $\{m + \ell-1, n\}$ desired closed-loop poles are arbitrarily assignable by the Davison-Wang method. This means that if $m + \ell \leq n$, then not all n poles of the system (2-12) - (2-13) can be replaced by desired ones.

(2) However, not all n-dimensional systems can have as many as min $\{m + \ell-1, n\}$ poles replaced by desired ones. The reason is that assumptions (4-20) and (4-21) on the matrices B and C may not be satisfied: either the $(n-\ell + 1) \times m$ matrix $\bar{B}_2$ does not have full rank, or some column of the $\ell \times (n-\ell + 1)$ matrix $\bar{C}_2$ is a linear combination of the columns of the $\ell \times (\ell-1)$ matrix $\bar{C}_1$.

The existence of a vector θ satisfying conditions (4-22) - (4-23) is based on assumption (4-21). For many spacecraft systems, it is impossible to determine such a vector θ [8]. When no such vector θ exists, the desired system poles will not be protected from change during the third step of pole assignment using the Davison-Wang algorithm, and the maximum number of closed-loop poles assignable will be max $\{m,\ell\}$.

(3) The closed-loop poles actually assigned may not be exactly equal to, but only arbitrarily close to, the desired poles. The reason is that the $\ell \times \ell$ matrix product $CQ\hat{Q}H$ in (4-17) may not be invertible. Davison suggested that the closed-loop poles to be assigned be varied slightly so that the matrix product becomes nonsingular [1]. Alternately, the pseudo-inverse of this matrix product, in case it is not invertible, may be used. With such a procedure, the assigned closed-loop poles would be an approximation to the desired poles.

4.2.2.2 Numerical Difficulties

Numerical difficulties may arise in calculating the coefficients $a_1,\ldots,a_n$ by Eq. (4-8) and also in solving (4-17) for the gains $g_1,\ldots,g_\ell$. This is because the controllability matrix Q defined by (4-7) may be highly ill conditioned (even though it is nonsingular) when the dimension n of the system is large. Davison and Chow [3] suggested that a different time scale $\tau = t/\gamma$ be used so that $\|b\| \simeq \| \gamma^{n-1} A^{n-1} b\|$; i.e., the first and the last columns of the new controllability matrix, with A replaced by γA, have approximately the same magnitude.

The time scaling was successful for an example of a 41-dimensional system [3]. But, according to Mahn's experience with spacecraft systems [8], in no case did this suggestion improve the assignment of the poles, and in some cases it significantly affected the assignment adversely.

4.2.2.3 A Pitfall

In assigning max $\{m,\ell\}$ or min $\{m + \ell-1,n\}$ closed-loop poles, nothing can be said about the remaining poles [1]. In the course of this study, it was discovered that the real parts of the remaining poles may turn out to be positive while desired values are assigned to replace other less desirable poles by an output feedback control. (For an illustration, see Section 4.3.8.) A pitfall may exist, particularly when the remaining poles were considered desirable in the open-loop system and therefore left alone: potential instability of the closed-loop system may be left unnoticed.

Blind application of this design method may allow hidden instability to exist in the closed-loop system.

4.2.2.4 Other Weaknesses

(1) The $m \times \ell$ gain matrix G is restricted to have rank at most one (because of the assumed dyadic form). Design freedom is thus greatly reduced, since the general $m \times \ell$ gain matrix can have rank min $\{m,\ell\}$, which is usually much larger than 1.

(2) As a consequence of the dyadic form, the required feedback gains G_{ij} are usually very high, and the closed-loop system modes are highly coupled.

(3) The design may be quite non-optimal since no performance index on accuracy, control energy, or control time is ever considered for optimization in the design process.

4.2.3 Maturity

As a whole, methods for designing output feedback controllers by pole assignment have not yet reached maturity. Various modifications or extensions keep appearing in the literature; see for example References [9] to [17]. None have yet significantly advanced the state of the science, however.

Specific improvement on the Davison-Wang method is also required. In the course of this study, it was found that with some appropriate modifications on this method:

(1) It is possible to predict, before actually computing the output-feedback gain matrix for assignment of desired closed-loop poles, whether hidden instability in the resultant closed-loop system may exist; i.e., it is possible to predict whether real parts of the remaining poles may become positive;

(2) It is possible, under certain conditions, to assign the n closed-loop poles with a given number ℓ of outputs, and these conditions can be checked by the computer before proceeding to compute the required gain matrix; and

(3) It is possible to compute the characteristic coefficients $a_1,\ldots,a_n$ without using Eq. (4-8) and hence to avoid the numerical difficulties with the matrix Q and its inverse Q^{-1}.

The details of these new findings and other possible improvements will be reported later.

4.2.4 Applicability to Large Flexible Space Structures

Since the fundamental modal design model (2-10) - (2-11) of a large flexible space structure has N modes, its state-space representation (2-14) - (2-15) has 2N poles (i.e., n = 2N). As mentioned in Section 2.2.5.2, for the $2N \times m$ matrix $B_C = \begin{bmatrix} 0 \\ \Phi_C^T B_A \end{bmatrix}$ to have rank m, it is necessary that $m \leq N$. Similarly, for the $\ell \times 2N$ matrix $C_C = [0, C_V\Phi_C]$ to have rank ℓ when only velocity sensors are used, or for the $\ell \times 2N$ matrix $C_C = [C_P\Phi_C, 0]$ to have rank ℓ when only position sensors are used, it is necessary that $\ell \leq N$. Consequently,

$$\min\{m + \ell-1, n\} = \min\{m + \ell-1, 2N\} = m + \ell-1 \leq 2N-1.$$

This means that it is never possible to assign desired values to all 2N closed-loop poles by the Davison-Wang method, if only velocity sensors, or only position sensors, are used. Then, at least one pole will always have to be left alone. This limits the designer's ability to actually alter all the fundamental characteristics of the structure as desired. For an illustration, see Section 4.3.1.

Moreover, hidden instability may exist in the closed-loop system, since the real parts of the remaining poles may turn out to be positive. For a discussion on the pitfall, see Section 4.2.2.3; for an illustration, see Section 4.3.8.

For all 2N closed-loop poles of the fundamental state-space design model (2-14) - (2-15) to be assignable by the Davison-Wang method, the ℓ sensors must include at least one velocity sensor and at least one position sensor. The total number of velocity and position sensors required for assigning all 2N poles by the Davison-Wang method must be at least 2N-m + 1. Locating position sensors away from velocity sensors makes it easy for the $\ell \times 2N$ matrix $C_C = [C_P\Phi_C, C_V\Phi_C]$ to have rank at least N + 1.

For the Davison-Wang method to be applicable to a large space structure, the location and the number of actuators placed on the structure must be chosen so that the critical modes are completely controllable. Similarly, the location and the number of sensors placed on the structure must make the critical modes completely observable. See Appendix B for discussions on complete controllability and complete observability of critical modes and an algorithm for determining the proper location and proper number of actuators and sensors.

As mentioned in Section 4.2.2.3, the Davison-Wang method does not guarantee the closed-loop stability even of the (reduced-order) model on which the design of output-feedback controller is based. (For an illustration, see Section 4.3.8.) The closed-loop stability of the large finite-element model (2-4) - (2-5) with a feedback controller based on a reduced-order design model (2-10) - (2-11) is even more questionable, let alone the closed-loop stability of the actual infinite-dimensional distributed-parameter flexible structure.

The relative simplicity in the design and implementation of an output-feedback controller, compared with the combination of a state-feedback controller and an observer, may permit more than the critical modes to be included in the design model. However, with a large flexible space structure, which is infinite-dimensional in nature, there are still many modes that cannot be included. Model errors due to truncation as well as roundoff are inevitable. So far, the Davison-Wang method has no provision for guaranteeing robustness against model errors, parameter variations, control spillover, or observation spillover.

Control spillover and observation spillover in an output feedback loop do not necessarily destabilize the large structure, though closed-loop performance of the output-feedback controller may be drastically degraded thereby. For an illustration, see Section 4.3.3.

The Davison-Wang method was recently applied to two idealized spacecraft with flexible appendages [8], [17]. It was concluded that, although it still required extensive investigation in order to resolve many practical difficulties, the Davison-Wang pole assignment process could be a viable preliminary control design tool [17].

4.3 Illustration

The simple test problem described in Section 2.5 is taken as an example for illustration of the Davison-Wang method of pole assignment. The eight points of interest (questions) listed also in Section 2.5.3 are addressed as follows.

4.3.1 Question 1

The Davison-Wang method is not applicable to the fundamental state-space design model (2-25)-(2-26) as is, since it is impossible to satisfy the assumption that the 2 × 2 matrices

$$B = \begin{bmatrix} 0 \\ \hline \phi_2^T \end{bmatrix} \text{ and } C = [0 \vdots \phi_2], \quad \text{with } \phi_2 = \begin{bmatrix} -0.857 \\ 0.365 \end{bmatrix}$$

have rank 2. Note that $N = 1$, $m = \ell = 2$, as is.

The number of independent actuators and of independent velocity sensors must be reduced so that, for the new numbers m and ℓ, the resultant $2N \times m$ matrix B has rank m and the $\ell \times 2N$ matrix C has rank ℓ. This can be achieved by combining the actuator inputs and sensor outputs, respectively. The following special combination is considered in this illustration: only actuator 2 and sensor 2 are used. Therefore, the fundamental state-space design model (2-25) - (2-26) becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6.702 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.365 \end{bmatrix} u_2 \qquad (4\text{-}26)$$

$$y_2 = [0 \quad 0.365] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (4\text{-}27)$$

From here on, $m = 1$, $\ell = 1$, $n = 2$;

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \eta_2 \\ \dot{\eta}_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -6.702 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.365 \end{bmatrix}, \quad C = [0,\ 0.365]$$

Since $p \triangleq \min\{m + \ell - 1, n\} = 1$, at most one desired closed-loop pole can be assigned by the Davison-Wang method. Furthermore, since $\max\{m,\ell\} = 1 = \min\{m + \ell - 1, n\}$, this closed-loop pole is to be assigned in one step, using Davison's construction algorithm. Note that the system (4-26)-(4-27) has two poles, but only one of them can be re-assigned.

Davison's construction algorithm, as summarized in Section 4.1.3.1.6, is now used to assign one desired closed-loop pole to system (4-26)-(4-27).

Steps 1-5: Since the system is already a single-input system, steps 1-3 can be skipped. The number θ in step 4 and the vector b in step 5 are given by

$$\theta = 1, \qquad b = B = \begin{bmatrix} 0 \\ 0.365 \end{bmatrix}$$

Step 6:

$$Q = [b, Ab] = \begin{bmatrix} 0 & 0.365 \\ 0.365 & 0 \end{bmatrix}$$

Consequently,

$$Q^{-1} = \frac{1}{0.365} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Step 7: $$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = Q^{-1}A^n b = Q^{-1}A^2 b = \begin{bmatrix} -6.702 \\ 0 \end{bmatrix}$$

Step 8: $$\hat{Q} = \begin{bmatrix} -a_2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Q\hat{Q} = 0.365 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore,

$$CQ\hat{Q} = [0,\ 0.365^2]$$

Step 9: Let the desired closed-loop pole be denoted by parameter λ_1, which must be negative-valued, and postpone the discussion on its value. Substituting λ_1 in (6-15)-(6-16) yields

$$\Lambda = \Lambda_1 = \lambda_1^2 + 6.702$$

$$H = h_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$$

Step 10: $CQ\hat{Q}H = [0\ 0.365^2]\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} = 0.365^2\ \lambda_1$

$$(CQ\hat{Q}H)^{-1} = \frac{1}{0.365^2\lambda_1} = \frac{1}{0.133\lambda}$$

Step 11: $g^T = \Lambda\ (CQ\hat{Q}H)^{-1} = (\lambda_1^2 + 6.702)/0.133\lambda_1$

Step 12: $G = \theta g^T = g^T = \dfrac{\lambda_1^2 + 6.702}{0.133\lambda_1}$ (4-28)

therefore

$$u_2 = G\ y_2 = \frac{\lambda_1^2 + 6.702}{0.133\lambda_1}\ y_2 \tag{4-29}$$

This test problem calls for at least 10% of critical damping on mode 2. Substituting (6-29) into the fundamental state-space design model (6-26)-(6-27) yields the following closed-loop system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6.702 & \dfrac{\lambda_1^2 + 6.702}{\lambda_1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4-30}$$

The closed-loop poles are

$$\lambda_1,\ \frac{6.702}{\lambda_1}$$

Note that one of the closed-loop poles is exactly given by the parameter λ_1, while the other is inversely proportional to λ_1. Figure 4-3 shows the position and variation of these two poles as $|\lambda_1|$ increases until $\lambda_1 = -2.588$. In contrast, Figure 4-4 shows the position and variation as $|\lambda_1|$ decreases until $\lambda_1 = -2.588$. It is not difficult to conclude that the system (4-30) is critically damped if and only if $\lambda_1 = -2.588$, but overdamped if and only if $\lambda_1 \neq -2.588$. In other words, according to the fundamental design model (2-23)-(2-24), or equivalently (4-26)-(4-27), mode 2 will never be underdamped with feedback control given by (4-29).

Since all damping will be more than 10C% of critical damping, it is desirable to choose

$$\lambda_1 = -2.588 \tag{4-31}$$

for critical damping of mode 2. Consequently, from (4-23) and (4-29),

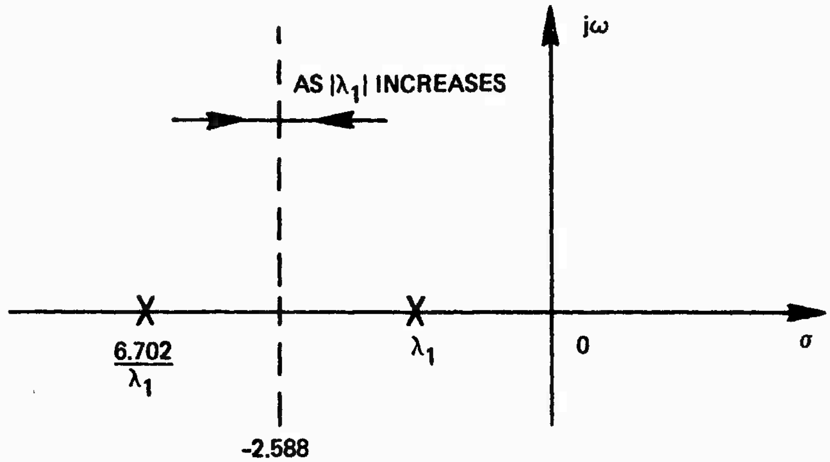


Figure 4-3. Position and variation of the two poles for $-2.588 \leq \lambda_1 \leq 0$.

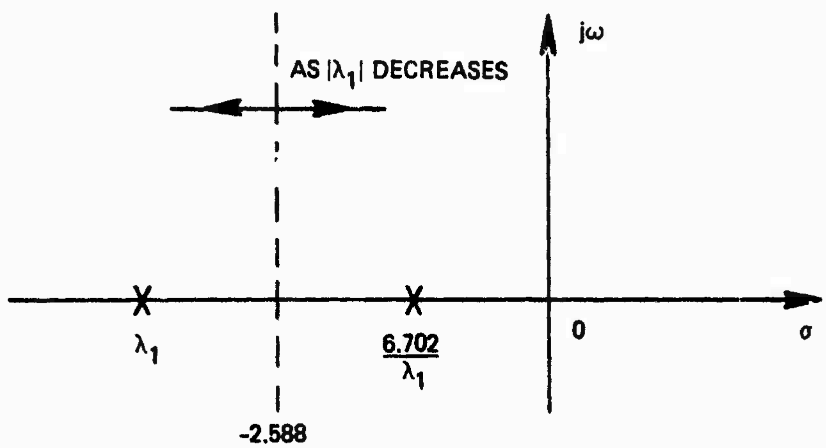


Figure 4-4. Position and variation of the two poles for $\lambda_1 \leq -2.588$.

$$G = -38.917 \qquad (4\text{-}32)$$

$$u_1 = 0, \quad u_2 = -38.917\, y_2 \qquad (4\text{-}33)$$

4.3.2 Question 2

With control given by (4-33), the closed-loop system of the finite-element modal dynamic model (2-21)-(2-22) is

$$\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 14.292 & 8.608 \\ 8.608 & 5.185 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 0.298 & 0 \\ 0 & 6.702 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = 0 \qquad (4\text{-}34)$$

The closed-loop poles are

$$\begin{array}{l} -0.021 \\ -0.041 \quad \pm j2.240 \\ -19.356 \end{array}$$

Since all these four poles have strictly negative real parts, the closed-loop system is asymptotically stable.

Figure 4-5 shows the configuration of the open-loop and closed-loop poles. The closed-loop poles no longer have the same natural frequencies as the open-loop poles; the identity of the modes is thus lost. Nevertheless, the pair $-0.041 \pm j2.240$ of closed-loop poles may be identified as the shifted mode 2 because of its proximity to the pair $\pm j2.589$ of open-loop poles, which corresponds to mode 2. Consequently, the other pair, -0.021 and -19.356, may be identified as the shifted mode 1. With such identification, it is then possible to answer the second question posed. The two modes now have damping ratios

$$\zeta_1 = 1535\%$$
$$\zeta_2 = 1.822\%$$

respectively.

Note that mode 2 was designed to have 100% of critical damping, but now it actually has only 1.822%, 5.5 times less the minimum required.

4.3.3 Question 3

Rewrite Eq. (4-34) in component form:

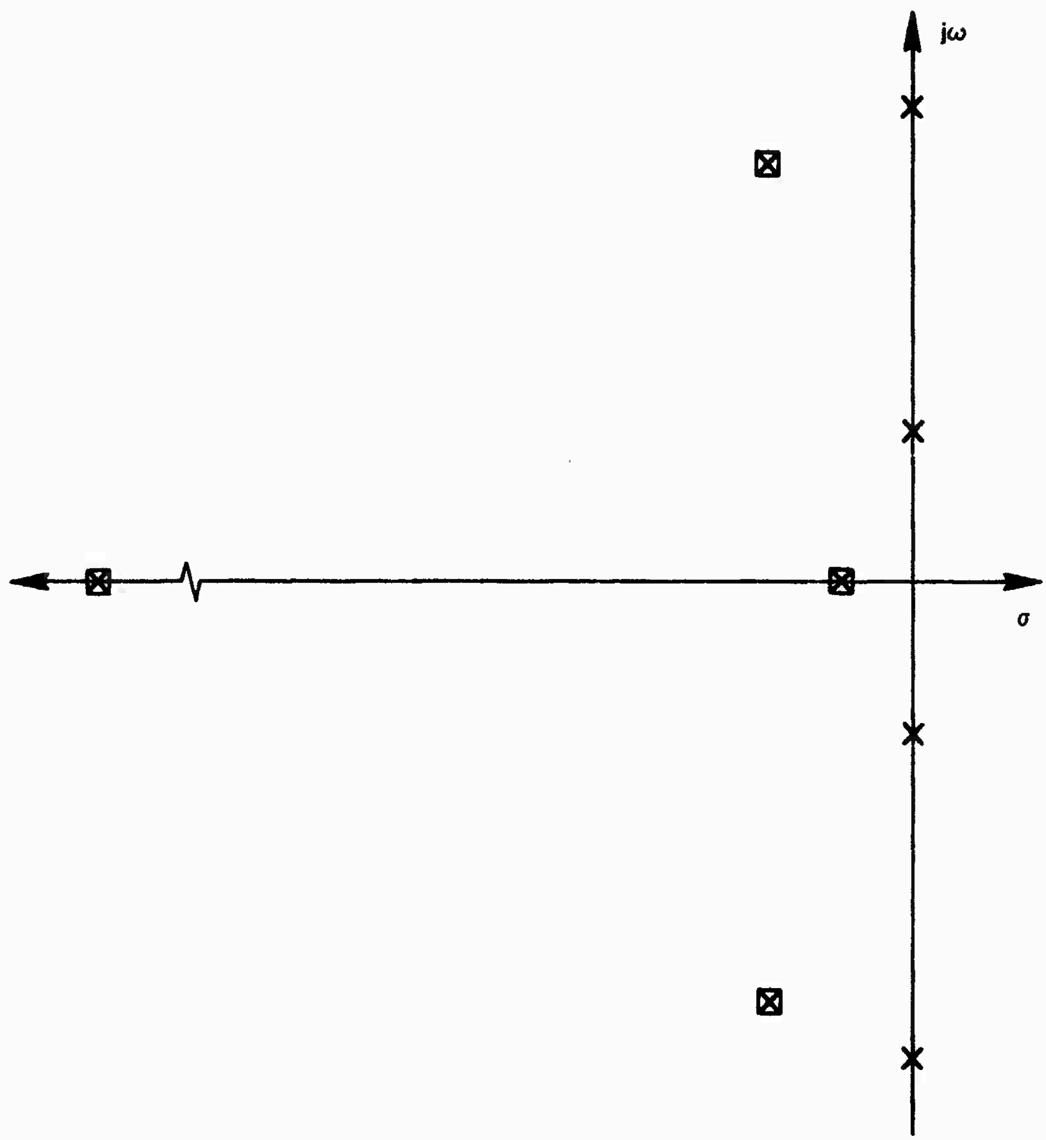


Figure 4-5. Configuration of the poles, open-loop poles (X): ±j0.546, j2.589, closed-loop poles ($\boxed{X}$): -0.021, -19.356, -0.041 ± j2.240.

$$\ddot{\eta}_1 + \underbrace{14.292\,\dot{\eta}_1}_{\text{C.S., O.S.}} + 0.298\,\eta_1 \qquad \underbrace{+8.608\,\dot{\eta}_2}_{\text{C.S.}} = 0 \tag{4-35}$$

$$\ddot{\eta}_2 + 5.185\,\dot{\eta}_2 + 6.702\,\eta_2 \qquad \underbrace{+8.608\,\dot{\eta}_1}_{\text{O.S.}} = 0 \tag{4-36}$$

The term 8.608 $\dot{\eta}_1$ in (4-36) results from observation spillover. It causes the designed control performance on the critical mode (mode 2) to degrade. In particular, the designed 100% of critical damping on mode 2 is thereby reduced to only 1.822%, as computed in Section 4.3.2.

The term 8.608 $\dot{\eta}_2$ in (4-35) results from control spillover. It excites the residual mode (mode 1) whenever the critical mode (mode 2) is in motion. This indirectly degrades the designed control performance on the entire system.

Finally, the term 14.292 $\dot{\eta}_1$ is a product of two factors which characterize both control spillover and observation spillover, respectively. It gives extremely heavy and unexpected damping to the residual mode. As computed in Section 4.3.2, the damping on the residual mode is increased from null to 1535% of critical damping. No such strong damping would exist if either control spillover or observation spillover were completely eliminated.

Observe that if there were no observation spillover but control spillover, mode 2 would have the designed critical damping, but mode 1 would have no damping at all and would vibrate forever once it is excited because of control spillover or any other disturbance.

Observe also that if there were no control spillover but observation spillover, designed control performance on mode 2 would be degraded, and mode 1 would still have no damping (as planned) and would not be excited due to control spillover. Mode 1 still might vibrate forever if it were excited by other disturbances.

4.3.4 Question 4

Figure 4-6 shows the time responses $q_1(t)$ and $q_2(t)$. The peak magnitude of $q_2(t)$ is equal to 1 meter at t = 0. The 5% settling time of $q_2(t)$ is approximately 140 seconds. The steady state of $q_2(t)$ is zero. Similar characterization can be made of the time response $q_1(t)$. The output-feedback controller designed by the Davison-Wang method does suppress the vibration in both masses.

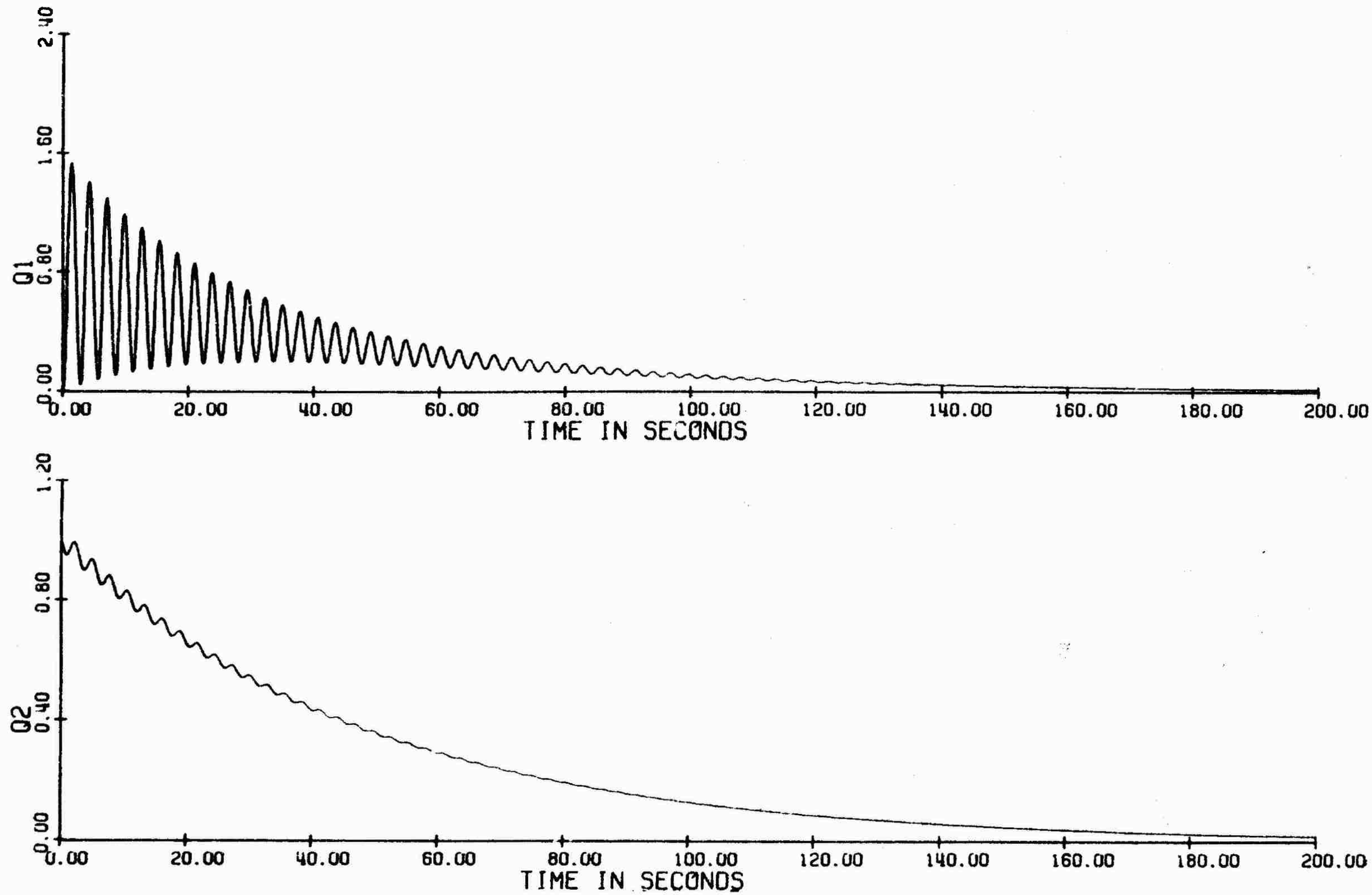


Figure 4-6. Closed-loop time response with $q_2(0) = 1$.

4.3.5 Question 5

Figure 4-7 shows the closed-loop time responses $q_1(t)$ and $q_2(t)$. The peak magnitude of $q_2(t)$ is approximately 0.017, compared with 0.4 in the open-loop case as shown in Figure 2-5. In fact, mass 2 remains vibrating in the steady state, but with amplitude only approximately 0.008, a reduction of 50 times.

4.3.6 Question 6

Note that the feedback control (4-33) is a special case of the following parametric form

$$u_1 = 0, \quad u_2 = -k\, y_2 \tag{4-37}$$

Substituting it in the finite-element discrete dynamic model (2-19)-(2-20) yields

$$\begin{cases} \ddot{q}_1 + 5q_1 + 4q_2 = 0 & \text{(4-38)} \\ 2\ddot{q}_2 + k\dot{q}_2 + 4q_2 + 4q_1 = 0 & \text{(4-39)} \end{cases}$$

An analysis of the root locus for system (4-38), (4-39) answers the above questions. First of all, the characteristic equation is

$$2s^4 + ks^3 + 14s^2 + 5ks + 4 = 0$$

Hence

$$1 + \frac{ks(s^2 + 5)}{2(s^4 + 7s^2 + 2)} = 0 \tag{4-40}$$

The open-loop poles and zeros associated with Eq. (4-40) are

poles: $\pm j0.546, \pm j2.589$

zeros: $0, \pm j2.236$

A sketch of the root locus as k increases from zero to infinity is shown by Figure 4-8. Observe that as k increases from zero to infinity, the damping ratio of mode 1 also increases from zero to infinity but that of mode 2 increases from zero to some value corresponding to k* then decreases to zero. An obvious answer to the question is: the damping of both modes may be simultaneously increased, but there are limitations. If $k^* \le -G = 38.917$ as given by (4-32), then the answer to the first question is no, that is, damping on both modes cannot be further increased.

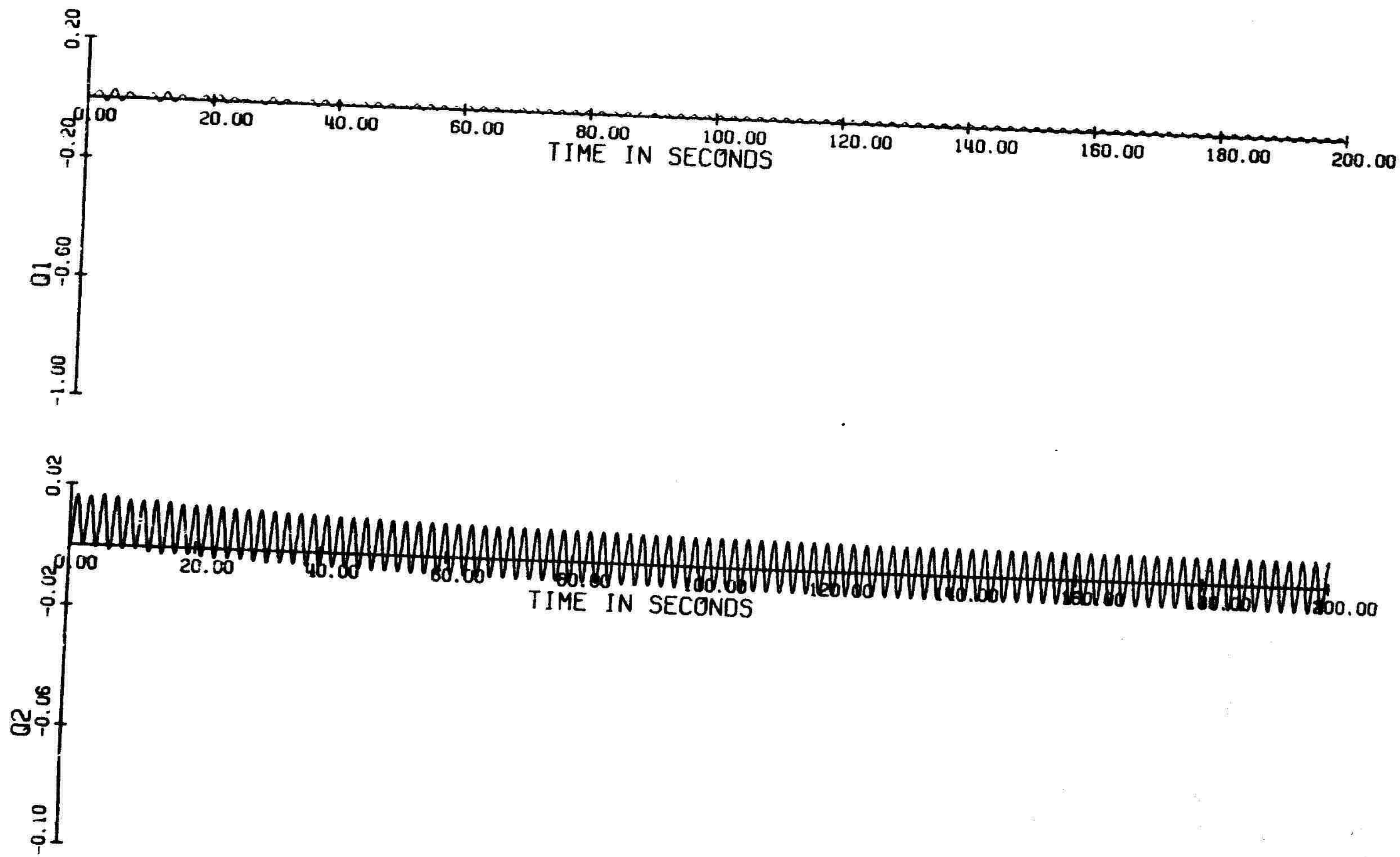


Figure 4-7. Closed-loop time response with $f_2(t)$ = sin 3t.

Since the Davison gain is given by (4-28), the parameter k is accordingly given by $k = \left(\lambda_1^2 + 6.702\right)/-\lambda_1$ for $\lambda_1 \leq 0$. Note first that as $|\lambda_1|$ increases from zero to infinity, k decreases from infinity to k = 38.917 at $\lambda_1 = -2.588$, then increases to infinity. This means that using a value for λ_1 other than -2.588 may further increase damping on both modes, provided that $k^* > 38.917$.

4.3.7 Question 7

For the Davison-Wang method to be applicable, the number of actuators has been reduced from two to one, and so has the number of velocity sensors. See Section 4.3.1. The number cannot be further reduced. The minimum is one actuator and one sensor.

4.3.8 Question 8

With position sensors instead of velocity sensors, Eq. (2-20) should be replaced by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

Equations (2-22), (2-24), and (2-26) should all be appropriately replaced. Specifically, (2-26) should be replaced by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = [\phi_2 \;\; 0] \begin{bmatrix} \eta_2 \\ \dot{\eta}_2 \end{bmatrix} \tag{4-41}$$

Using the same arguments as in Section 4.3.1, only actuator 2 and only position sensor 2 are used. The corresponding fundamental state-space design model is, from (2-25) and (4-41)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6.702 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.365 \end{bmatrix} u_2 \tag{4-42}$$

$$y_2 = [0.365 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4-43}$$

Equations (4-42) - (4-43) are the same as (4-26) - (4-27) except that now C = [0.365, 0].

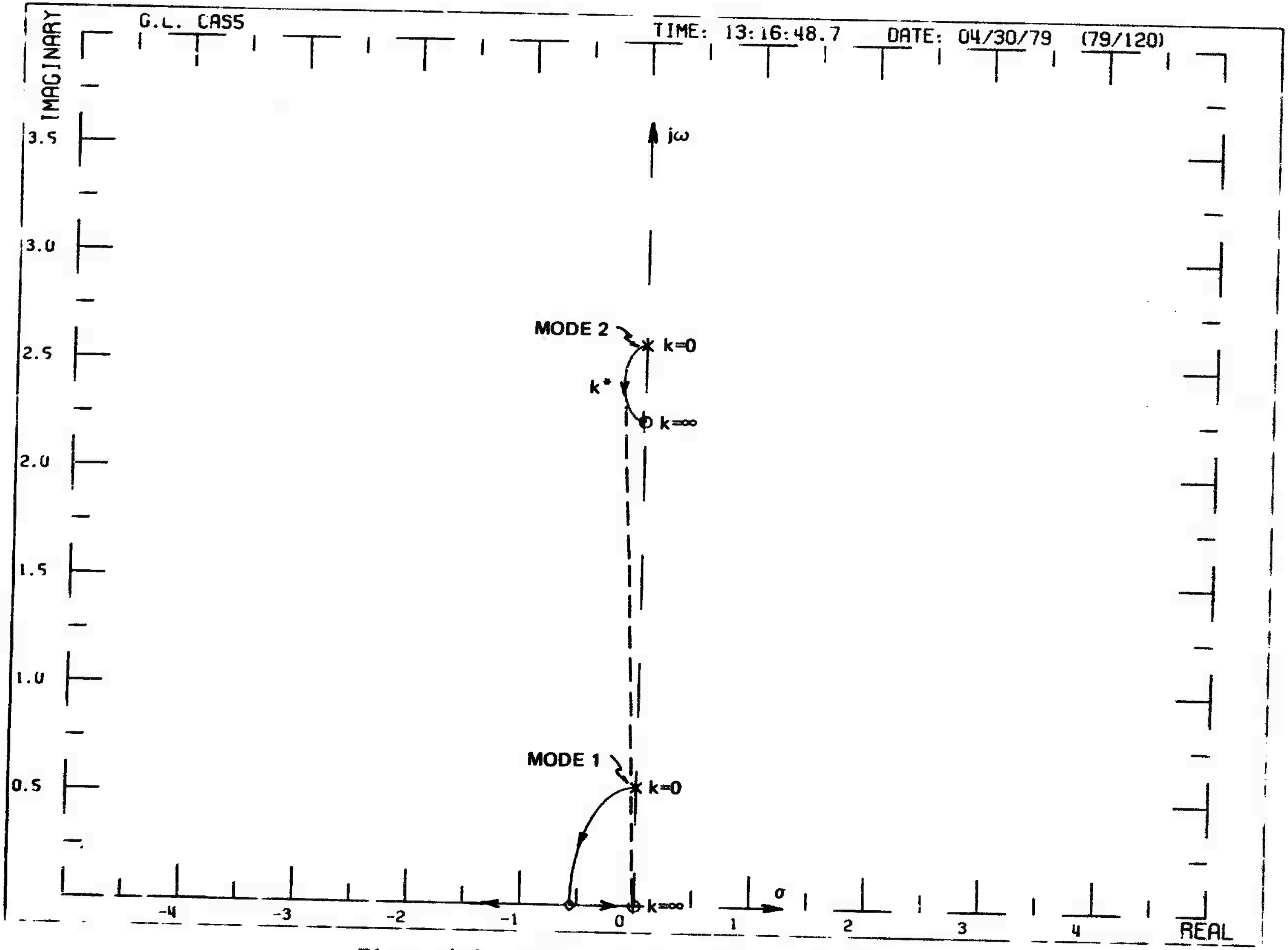


Figure 4-8. The root locus of Eq. (4-40).

Following the same steps as in Section 4.3.1, one obtains at the end of step 12 the following output feedback control

$$u_2 = \frac{\lambda_1^2 + 6.702}{0.133} y_2 \tag{4-44}$$

Note that only one pole, λ_1, can be assigned. Also recall that nothing can be said about the remaining pole.

Substituting (4-44) into (4-42) and (4-43) yields the following closed-loop system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \lambda_1^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4-42}$$

The closed poles are: λ_1 and $-\lambda_1$. Note that $-\lambda_1$ denotes the remaining pole, which becomes strictly positive when λ_1 is assigned a strictly negative value. The closed-loop system of model (4-42)-(4-43) is unstable! The remaining pole, about which nothing can be said by the Davison-Wang method, causes the instability.

4.4 Conclusions

4.4.1 Summary of Advantages

(1) The method enables the designer to have direct control over the modes of system response.

(2) The approach is modern, analytical, and systematic; its concept is simple; and its algorithm is straightforward.

(3) The dyadic form of the output-feedback gain matrix is simple to implement.

(4) The size of the computer memory and the amount of processing time required in the design process are much less than that required for the solution of Riccati equations.

(5) The controller can be implemented by electronic or electromechanical hardware, as well as computer software.

(6) The controller is much more robust to model errors and parameter variations than a corresponding controller using a dynamic estimator. No reliance on onboard computer is necessary.

4.4.2 Summary of Disadvantages

(1) There are theoretical limitations: not all system poles can be replaced by desired poles; the closed-loop poles actually assigned may not be exactly what are desired.

(2) There are numerical difficulties: the calculation of characteristic coefficients and the solution of feedback gains have numerical difficulties due to ill-conditioning of the controllability matrix Q.

(3) A pitfall exists: blind application of this method may allow hidden instability to exist in the closed-loop system.

(4) There are other weaknesses: design freedom is lost because of the restriction to a dyadic form of gain matrix; the required feedback gains are usually high; closed-loop system modes are highly coupled; the design may be quite nonoptimal with respect to a performance index on accuracy, control energy, or control time.

(5) The method is not yet mature: various modifications or extensions keep appearing in the literature; several possible improvements have been found during this study.

4.4.3 Final Comments

(1) Large flexible space structures have some special problems in the application of this method, as listed and explained in Section 4.2.4. These problems are not very serious or extraordinary, but do require attention.

(2) The use of the test problem given in Section 2.5 has given many valuable insights into the application of this method to the control of large flexible structures. However, tests on typical large flexible space structures are still needed for making realistic assessment.

(3) The Davison-Wang method can be a viable tool for preliminary or prototype design of active control systems for large flexible space structures, but it requires extensive further research to realize its many advantages over other methods. Preliminary findings, as outlined in Section 4.2.3, show that many of its disadvantages can be eliminated or reduced without compromising it advantages.

References

1. Davison, E.J., "On Pole Assignment in Linear Systems with Incomplete State Feedback," IEEE Trans. Automatic Control, Vol. AC-15, pp. 348-351, June 1970.

2. Davison, E.J., and Chatterjee, R., "A Note on Pole Assignment in Linear Systems with Incomplete State Feedback," IEEE Trans. Automatic Control, Vol. AC-16, pp. 98-99, February 1971.

3. Davison, E.J., and Chow, S.G., "An Algorithm for the Assignment of Closed-Loop Poles Using Output Feedback in Large Linear Multivariable Systems," IEEE Trans. Automatic Control, Vol. AC-18, pp. 74-75, February 1973.

4. Davison, E.J., and Wang, S.H., "On Pole Assignment in Linear Multivariable Systems Using Output Feedback," IEEE Trans. Automatic Control, Vol. AC-20, No. 4, pp. 516-518, August 1975.

5. Brasch, F.M., and Pearson, J.B., "Pole Placement Using Dynamic Compensators," IEEE Trans. Automatic Control, Vol. AC-15, No. 1, pp. 34-43, February 1970.

6. Davison, E.J., and Wang, S.H., "Properties of Linear Time-Invariant Multivariable Systems Subject to Arbitrary Output and State Feedback," IEEE Trans. Automatic Control, Vol. AC-18, No. 1, pp. 24-32, February 1973.

7. Topaloglu, T., and Seborg, D.E., "An Algorithm for Pole Assignment Using Output Feedback," in Proc. 1974 Joint Automatic Control Conf., pp. 309-312, June 1974.

8. Mahn, R.H., Jr., "Algorithm for and Implementation of Pole Assignment Using Output Feedback in Linear Multivariable Time-Invariant Systems," M.S. Thesis, Massachusetts Institute of Technology, June 1977.

9. Munro, N., "Further Results on Pole-Shifting Using Output Feedback," in Control System Design by Pole-Zero Assignment, Fallside, F., Ed. London: Academic Press, 1977 (based on papers presented at a Working Party at Cambridge University, September 1974), pp. 123-135.

10. Power, H.M., "New Approaches to Eigenvalue Assignment by Output Feedback and Numerical Control by State Feedback," ibid, pp. 181-185.

11. Seraji, H., "On Output Feedback Control of Linear Multivariable Systems," ibid, pp. 196-210.

12. Sridhar, B., "Investigation of Rank 2 and Higher Output Feedback for Pole Placement," ibid, pp. 210-225.

13. Kimura, H., "Pole Assignment by Gain Output Feedback," IEEE Trans. Automatic Control, Vol. AC-20, No. 4, pp. 509-516, August 1975.

14. Kimura, H., "A Further Result on the Problem of Pole Assignment by Output Feedback," IEEE Trans. Automatic Control, Vol. AC-22, No. 3, pp. 458-463, June 1977.

15. Porter, B., "Eigenvalue Assignment in Linear Multivariable Systems by Output Feedback," Int. J. Control, Vol. 25, No. 3, pp. 483-490, 1977.

16. Seraji, H., " A Note on Pole Assignment with Output Feedback," Int. J. Control, Vol. 27, No. 1, pp. 141-142, 1978.

17. Tseng, G.T., and Mahn, R.H. Jr., "Flexible Spacecraft Control Design Using Pole Allocation Technique," J. Guidance and Control, Vol. 1, No. 4, pp. 279-281, July-August 1978.

SECTION 5

OPTIMAL OUTPUT FEEDBACK CONTROL VIA LEVINE-ATHANS METHOD

5.1 Introduction

This section discusses the optimal output feedback control algorithm developed by Levine, Athans, and Johnson [1,2] and its application to designing vibration controllers for large space structures (LSS).

The problem studied by Levine, Athans, and Johnson is the following. One is given a continuous, linear plant which is modelled in state vector form

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) \tag{5-1}$$

$$\underline{x}(t_o) = \underline{x}_o \tag{5-2}$$

$$\underline{y}(t) = C\underline{x}(t) \tag{5-3}$$

where $\underline{x}(t)\varepsilon R^n$, $\underline{u}(t)\varepsilon R^m$, and $\underline{y}(t)\varepsilon R^r$. The symbols A, B, and C designate matrices, of appropriate dimensions, whose elements are time-invariant and known. $\underline{y}(t)$ is the output vector; that is, the plant is assumed to include a sensing system which measures $\underline{y}(t)$. The assumptions are made that: (1) $r < n$, and (2) the $r\times n$ matrix C has full rank. The plant is to be regulated using constant gain output feedback

$$\underline{u}(t) = -F\underline{y}(t) \tag{5-4}$$

The problem is to determine values for the elements of the $m\times r$ time-invariant gain matrix F.

The technique investigated by Levine, Athans, and Johnson seeks to establish F optimally using the usual infinite-time quadratic loss function

$$J(F) = \frac{1}{2}\int_{t_o}^{\infty} [\underline{x}^T(t)Q\underline{x}(t) + \rho\underline{u}^T(t)N\underline{u}(t)]dt \tag{5-5}$$

where $Q = Q^T \geq 0$, $N = N^T > 0$, and $\rho > 0$

The optimization problem posed by equations (5-1) to (5-5) differs from the standard infinite-time linear quadratic (LQ) one solely in that the above problem makes the realistic assumption that only the output vector $\underline{y}(t)$ is available for feedback. In the standard LQ problem, the (usually) unrealistic assumption is made that the full state vector $\underline{x}(t)$ is available for use by the controller; that is, in the standard LQ problem, equation (5-4) is replaced by $\underline{u}(t) = -F\underline{x}(t)$. In order to implement a controller which assumes full state feedback, it usually is necessary to add into the control loop an observer

which will use the measurement data $\underline{y}(t)$ to generate an estimate, $\hat{\underline{x}}(t)$, of the true state response $\underline{x}(t)$. This approach is very common. With such a technique, design of the controller (i.e., determination of F) and of the observer is relatively easy because of the well-known principle of separation between estimation and control [3]. The optimal output feedback technique of Levine, Athans, and Johnson does not employ an observer; hence it requires less on-line computations. However, it is much more difficult to determine the optimal gains F in the Levine-Athans-Johnson problem than it is in the standard LQ problem; that is, the off-line computation requirements are greater, and successful results are less certain. Also, in general, the Levine-Athans-Johnson method will yield a larger value for the cost than is achievable with state feedback and hence, in principle at least, poorer dynamic performance.

Some restrictions on A, B, C, and Q in addition to the above-noted ones that $r < n$ and rank $C = r$ are required in order to be certain that the design problem specified by equations (5-1) to (5-5) is solvable. Analogous requirements in the LQ state feedback problem are: (1) that (A,B) be stabilizable, and (2) that (A,D) be detectable. Conditions (1) and (2) are sufficient for the existence of a unique stable solution to the state feedback problem [3]. The matrix D in condition (2) can be any matrix such that there exists a matrix $M = M^T > 0$ which yields $Q = D^T M D$. Condition (2) is more-commonly stated as the requirement that $(A, \sqrt{Q})$ be detectable; this requirement is satisfied automatically if Q is selected to be positive definite.

For the optimal output feedback problem, condition (1) above must be modified to the necessary requirement that (A,B,C) be stabilizable by output feedback. This requires that A, B, C are such that there exist matrices F which yield (A-BFC) asymptotically stable. It is known [4] that a necessary condition for stabilizability by output feedback is that all uncontrollable modes of (A,B) and all unobservable modes of (A,C) be stable. A sufficiency condition for stabilizability by output feedback has been reported by Li [4] and by Denham [5].

Derivation of the Levine-Athans-Johnson algorithm is straightforward and well-documented in readily-accessible literature [1,2]. Hence, a derivation will not be presented here. One point in the work should be mentioned, however. When one attempts to minimize J of equation (5-5), the gain F turns out to be a function of the initial state $\underline{x}_o$. (This does not occur in the standard state feedback LQ problem.) Use of an F which is a function of $\underline{x}_o$ is usually neither possible nor desirable. Levine, Athans, and Johnson circumvented this difficulty by regarding $\underline{x}_o$ as a random vector and setting up an algorithm for minimizing the expected value EJ rather than J itself. As a result, F-matrices obtained by their algorithm can only be optimal on the average.

The basic equations which were obtained by Levine, Athans, and Johnson are as follows

$$K[A - BFC] + [A - BFC]^T K + Q + \rho C^T F^T N F C = 0 \qquad (5\text{-}6)$$

$$L[A - BFC]^T + [A - BFC] L + X_o = 0 \qquad (5\text{-}7)$$

$$\frac{\partial(EJ)}{\partial F} = \rho NFCLC^T - B^T KLC^T \tag{5-8}$$

(They did not actually indicate equation (5-8) explicitly in [1] or [2].) The new terms above are EJ, X_o, K, and L. The scalar EJ is the expected value of J when $\underline{x}_o$ is treated as a random vector. The n×n matrix X_o is the covariance of $\underline{x}_o$. ($E\underline{x}_o$ is assumed to be 0.) The matrices K and L both are symmetric and n×n. The matrix K was developed by manipulating equations (5-1) to (5-5) to produce:

$$EJ = \frac{1}{2} \text{ trace } KX_o \tag{5-9}$$

where

$$K \triangleq \int_0^\infty e^{[A-BFC]^T\tau} \, [Q + \rho C^T F^T NFC] \, e^{[A-BFC]\tau} d\tau \tag{5-10}$$

Equation (5-6) is equivalent to equation (5-10) if the integral exists. The matrix L develops when EJ is differentiated to form $\partial(EJ)/\partial F$. The basic definition of L is

$$L \triangleq \int_0^\infty e^{[A-BFC]\tau} X_o e^{[A-BFC]^T\tau} \, d\tau \tag{5-11}$$

Equation (5-7) is equivalent to equation (5-11) when the integral exists.

When utilizing the Levine-Athans-Johnson approach, the problem is to utilize equations (5-6) to (5-8) to find the F which minimizes EJ in equation (5-9). The technique considered by Levine, Athans, and Johnson utilized the necessary condition $\partial(EJ)/\partial F = 0$. Assuming that equation (5-8) can be solved for F

$$F = \frac{1}{\rho} N^{-1} B^T KLC^T (CLC^T)^{-1} \tag{5-12}$$

However, the values of the elements of K and L are not known a priori. Hence, K and L must be calculated in conjunction with the calculation of F. Thus, the Levine-Athans-Johnson algorithm involves employing equations (5-6), (5-7), and (5-12) in an iterative procedure. In the remainder of this section, the term "Levine-Athans-Johnson algorithm" will be used to refer solely to this specific technique - the iterative use of equations (5-6), (5-7), and (5-12).

Other approaches to the solution of equations (5-6) to (5-8) have been described in the literature, and these will be summarized later. These methods generally involve rewriting F as a vector and employing one of the standard function minimization algorithms. In the remainder of this section, such algorithms will be referred to as mathematical programming algorithms.

Most of the approaches which have been reported in the literature involve a direct solution of equations (5-6) and (5-7) for K and L respectively. These are Lyapunov-type equations for which numerous solution algorithms are available.

Experience has indicated that solution of equations (5-6), (5-7), (5-12) (or, alternatively, (5-6), (5-7), (5-8)) is not easy. There are no known methods which guarantee convergence. Also, use of a very good initial estimate for F sometimes is necessary to guarantee that if convergence does occur it will be to the global minimum of J rather than merely to a non-minimum stationary point.

5.2 Discussion

5.2.1 Strengths and Weaknesses of the Levine-Athans-Johnson Algorithm

The general strengths are the following.

(1) Controllers designed using the algorithm have relatively simple on-board or on-line implementation requirements since they basically involve only constant gain feedback.

(2) The gains which are computed by the algorithm are optimal according to an infinite time quadratic criteria.

(3) The algorithm is relatively mature, having been the subject of numerous investigations over the past 7-8 years.

(4) The algorithm is certain to yield a stable system (assuming that the conditions on (A,B,C,Q) are met) except for difficulties which can arise due to imperfections in the design model of the plant.

The general weaknesses are the following.

(1) Computing the optimal gains can be difficult, unfeasible, or impossible. Convergence and the necessity of obtaining a good initial guess for the gains are the main problems.

(2) The computed gains are optimal only in a stochastic sense. That is, they are based on an average value of the initial condition $\underline{x}_o$.

(3) Few, if any, of the studies reported in the literature have considered realistic or really difficult controller design problems.

(4) The Levine-Athans-Johnson algorithm cannot handle constraints on or among the controller gains. (Mathematical programming algorithms, however, can handle such constraints.)

(5) Apparently little is known or can be said in general about the robustness of systems designed by this technique, or sensitivity to noise.

The question of strengths and weaknesses of the algorithm will be discussed again later in this section when considering its specific application to the LSS problem.

5.2.2 Literature Search Summary

The first study, by Levine and Athans, of the optimal constant gain output feedback design problem was published in 1970. Since that time numerous investigations of the problem and of the algorithm proposed by Levine and Athans have been presented in the literature. The following paragraphs attempt to summarize this material. It is not claimed that the articles noted below constitute a complete listing of the relevant published work.

The text by Anderson and Moore [6] includes a section on the optimal output feedback control problem. Equations equivalent to those (equations (5-6), (5-7), (5-12)) of Levine, Athans, and Johnson are presented. Two possible arrangements of these equations for iterative solution are listed and discussed. The authors stress that convergence cannot be guaranteed with either arrangement.

Knapp and Basuthakur [7] rederived the equations in [1,2] using an approach which the authors claimed to be mechanically simpler.

Choi and Sirisena [8] performed a computer study in which they compared the Levine-Athans-Johnson algorithms (equations (5-6), (5-7), (5-12)) with a method that used equations (5-6), (5-7), (5-8) (F represented as a vector); this second method employed the Davidon-Fletcher-Powell function minimization algorithm. A simple fourth order plant with two controls and three outputs was studied. The authors claimed enthusiastically that their work showed the Davidon-Fletcher-Powell method requires considerably less computation, that it appeared to ensure convergence, and that it should therefore make the design of optimal output feedback algorithms more viable. The authors noted that the class of problems that were being considered exhibit local minima and therefore the problem should be run several times using different initial values of F. They also claimed that their work indicates that if one starts with an initial value of F which yields (A-BFC) stable then it is not really necessary to test for stability of (A-BFC) on subsequent iterations.

The work of Bingulac, Cuk, and Calovic [9] indicated that the Levine-Athans-Johnson algorithm cannot guarantee satisfactory results, particularly when the number of outputs is much smaller than the order of the system. The problem, they claimed, is due to the inability of finding an adequate initial guess for F. They proposed to circumvent this difficulty by a technique which starts by solving the full state regulator problem (which does not require an initial guess of F) and then reducing the number of measurements in steps (with a new F being computed at each step) until the actual

desired output order r is attained. Their work assumed that $\underline{x}$ is transformed such that $C = [I_r | 0]$. They modified the basic equations of Levine, Athans, and Johnson into a significantly different form which they claimed to be computationally more convenient. (The motivation for this modification was not apparent.) A complete algorithm was developed. However, the details were not all presented in [9], nor did the authors indicate whether they had implemented and tested their algorithm in a computer program. No numerical results were presented.

Petkovski and Rakic [10] concurred that the Levine-Athans-Johnson algorithm cannot guarantee satisfactory results when the order of the system greatly exceeds the number of outputs and that the problem is due to the difficulty in obtaining an adequate initial guess of F. They proposed to surmount the problem by obtaining the initial guess of F through use of Kosut's minimum error excitation method [11]. Their work included a partial verification of their scheme by means of a very simple sample problem. In this problem, the dimension of $\underline{x}$ was four, and there were three inputs and three outputs. Convergence was achieved in nine passes.

Söderström [12] pointed out that there are (at least) two ways of solving the Levine-Athans-Johnson algorithm (equations (5-6), (5-7), (5-12)) iteratively. The techniques considered by Söderström are as follows:

Method 1

(1) Determine an initial K;

(2) Solve the nonlinear equations (5-7) and (5-12) for L and F;

(3) Solve the linear equation (5-6) for K;

(4) Repeat as necessary.

Method 2

(1) Determine an initial F;

(2) Solve the linear equations (5-6) and (5-7) for K and L;

(3) Compute F from equation (5-12);

(4) Repeat as necessary.

(These are the two arrangements noted in the Anderson and Moore text [6]; they also were noted in the Levine-Athans-Johnson papers.) Söderström was concerned with the stability of these two methods. He considered a trivial example with $n = 2$ and $m = r = 1$. The example was sufficiently simple that difference equations for the scalar gain f for each of the two methods could be derived

$$f_{k+1} = h(f_k) \quad , \quad k = 1,2,\ldots$$

Söderström was able to show analytically that Method 1 was convergent; Method 2, however, was locally-divergent for some X_o and q values. Söderström wisely did not attempt to draw general conclusions about the usual performance of the two methods from this single example.

Knox and McCarty [13] studied the problem of computing optimal output gains for aircraft flight control systems. In comparison to most of the other references examined in the present literature search, the work of Knox and McCarty was quite extensive and applications-oriented; it was the only study that considered sample problems that are in any way realistic. The work utilized equations equivalent to those ((5-6), (5-7), (5-8)) developed by Levine, Athans, and Johnson. However, it did not employ the solution-approach (equations (5-6, (5-7), (5-12)) which Levine et al proposed. Instead, Knox and McCarty considered only mathematical programming methods. Two techniques were developed and tested against the well known Davidon-Fletcher-Powell method. The problem which was considered involved the hypothetical design of a controller for the lateral-directional dynamics of the C-141 aircraft. The system was fourth order with two controls and three outputs. Q and N were chosen diagonal and positive definite. Good convergence results were achieved. In a second portion of the study, an algorithm which enables equality constraints to be placed on the elements of F was developed. (As noted earlier, the Levine-Athans-Johnson algorithm cannot include constraints on F.) This algorithm was tested in a problem in which the constraints were used to provide a system that would yield proper turn coordination. Again good results were obtained. The authors conceded, however, that their methods could be troubled by the local minima phenomena and that rerunning the problem using several sets of initial gains could sometimes prove necessary.

5.2.3 Applicability to Vibration Control of LSS

The general features, strengths, and weaknesses of the Levine-Athans-Johnson algorithm were noted in the preceding sections. The present section notes some additional considerations which are pertinent to the application of the algorithm to the specific problem of designing vibration controllers for LSS.

5.2.3.1 Strengths

1. The main strength of the algorithm in the LSS application is that it appears to have the potential to design controllers which can improve the damping of a large number of modes with a much smaller number of actuators and/or sensors. That is, the algorithm does not limit the designer to some prefixed and ironclad relation between the number of modes, actuators, and sensors.

2. The weighting terms Q, ρ, and N in the algorithm are selected by the designer. A wide variety of performance characteristics can be obtained depending on the choice of these terms.

3. The algorithm provides an approach for dealing with the residual modes which is not provided by the other output feedback controller design

techniques; namely, it includes in the design model (via A, B, C) as many modes as practical, but reduces the influence of the modes that one does not wish to control by weighting them only lightly, or not at all, in Q.

5.2.3.2 Weaknesses

1. Determining the optimal gain matrix F in LSS applications is usually a big problem; is usually a difficult problem; and may, in some applications, be an unfeasible or impossible problem. One of the difficulties is the size of the matrices which are involved. For example, a system with 32 actuators and 32 sensors would yield an F which is 32×32; in this case there would be 1024 gains to be determined. A computer program for determining this many gains is certain to be cumbersome, slow running, and expensive - possibly impractically so. Thus, there are limits (possibly undesirably low ones in LSS applications) on the size of the problems (i.e., on the values of m, n, r) to which the Levine-Athans-Johnson algorithm is amenable.

2. The studies reported in the literature indicate that obtaining a sufficiently accurate initial estimate of F can be a major difficulty and that the difficulty increases as the number of modes is increased relative to the number of sensors. This phenomena appears likely to limit the number of modes which can be included in the design model for a given number of sensors and actuators - thereby curtailing some of the potential advantages of the algorithm.

3. The fact that the Levine-Athans-Johnson algorithm cannot include constraints on F is regarded as a significant drawback of the technique in the LSS application. (It was noted earlier that such constraints can be included if mathematical programming methods are used to solve the equations.) For example, if the actuators and sensors are colocated, a controller with a diagonal F may be the best and most practical design in some LSS problems; the Levine-Athans-Johnson algorithm, however, cannot compute a diagonal F.

4. Apparently, little is known in general about the robustness which can be expected from controllers designed using the Levine-Athans-Johnson algorithm. Thus, in the LSS application, it is not possible at present to determine the extent to which controller performance can be degraded by modelling errors or by control and observation spillover.

5.2.3.3 Implementation Techniques and Considerations

1. With the exception of [9], the algorithms listed in the literature for solving the optimal output feedback equations (equations (5-6), (5-7), (5-8), or (5-6), (5-7), (5-12)) depend heavily on obtaining a numerical solution to equations (5-6) and (5-7). These are Lyapunov-type equations. Solving them numerically is not a trivial operation. Numerous algorithms for their solution, however, are available. Smith's method [14] appears capable of solving these equations in the LSS application if Smith's claims about the performance of his algorithm can be believed; a detailed study or literature search of Lyapunov-equation solving algorithms, however, was not performed in the present LSS work. In the LSS application, the matrices L and K must be

computed numerous times in each run. Both are n×n; this, therefore, is one of the factors that will place a practical upper limit on the dimension (n) of the state vector $\underline{x}$ which can be included in the design model when using the Levine-Athans-Johnson algorithm. When solving Lyapunov equations convergence tends to become more difficult if the damping is low [14]. Since damping normally will be small or nonexistent in LSS problems, this is one more factor which may increase the difficulties in applying the Levine-Athans-Johnson algorithm to LSS.

2. When solving any optimal output feedback problem, one must first choose values for the weighting terms Q, N, and ρ. One choice for Q is to select it such that $\underline{x}^T Q \underline{x}$ is proportional to a weighted sum of the mechanical energies in each vibration mode. It can be shown that this can be accomplished by defining Q as follows

$$Q = \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix} \begin{bmatrix} I_{n/2} & 0 \\ 0 & \Omega^2 \end{bmatrix}$$

where $\Omega^2 = \text{Diag.} [\omega_1 \; . \; . \; . \; \omega_{n/2}]$

$\Gamma = \text{Diag.} [\gamma_1 \; . \; . \; . \; \gamma_{n/2}]$

In the above expressions the ω's are the natural frequencies of the structural vibration modes, and the γ's are positive weighting factors assigned to each mode. $\underline{x}^T Q \underline{x}$ will be proportional to the system vibratory energy if the γ's are made unity. The above relation assumes that the state vector $\underline{x}$ is chosen as $\underline{x}^T = \{\dot{\underline{\eta}}^T, \underline{\eta}^T\}$ where the components of $\underline{\eta}$ are the instantaneous values of the structural modes. Inclusion of sensor or actuator dynamics in $\underline{x}$ necessitates only minor modifications to the above expression.

The most-readily apparent technique for selecting N is to use

$$N = \text{diag.} [w_1 \; . \; . \; . \; w_m]$$

where the w's are positive weighting factors assigned to each actuator. One might make all w's unity, unless there is a specific reason for doing otherwise.

Determining ρ a priori is more difficult. The usual procedure is to generate several controller designs, each obtained with a different ρ, and then to choose the controller whose performance is judged to be best.

3. When setting up a program for implementing the Levine-Athans-Johnson algorithm, it appears a practical necessity to include: (1) a routine to determine if (A-BFC) is asymptotically stable, and (2) a course of action to be followed if it is not.

4. The work to date indicates that the special features often encountered in LSS problems (zero structural damping, velocity sensing only, colocated sensors and actuators, negligible sensor and actuator dynamics) provide, at best, very limited potential for reducing the computational burden, or otherwise alleviating the difficulties, when establishing controller gains by the Levine-Athans-Johnson algorithm.

5. The studies reported in the literature have proven that use of a very good initial estimate, F_o, of F is crucial for successful application of the Levine-Athans-Johnson algorithm. The rudiments of a scheme for attacking this F_o problem is proposed below. This scheme is a blend of the sequential technique of Bingulac, Cuk, and Calovic (without their modification of the Levine-Athans-Johnson equations) and the single shot method proposed by Petkovski and Rakic which uses Kosut's minimum error criteria.

The steps are as follows:

(1) Add $q_{add} = (n - r)$ rows to $C(r \times n)$ to yield a $C_o(n\times n)$ which is nonsingular.

(2) Solve the optimal state feedback problem, using C_o, to obtain an $F_o(m\times n)$.

(3) Set $i = 1$.

(4) Choose q_i $\left(q_{i-1} < q_i \leq q_{add}\ ,\ q_o = 0\right)$
Delete the last q_i rows from C_o to obtain $C_i(r_i\times n)$ where $r_i = n-q_i$.

(5) Use Kosut's minimum error method to obtain $F_{INi}(m\times r_i)$

$$[A-BF_{i-1}C_{i-1}]\tilde{L} + \tilde{L}[A-BF_{i-1}C_{i-1}]^T + X_o = 0 \quad \text{(solve for } \tilde{L}\text{)},$$

$$F_{INi} = F_{i-1}C_{i-1}\tilde{L}C_i^T[C_i\tilde{L}C_i^T]^{-1}$$

(6) Use the Levine-Athans-Johnson algorithm to compute $F_i(m\times r_i)$

(a) $F_i = F_{INi}$

i=i+1

(b) $K[A-BF_iC_i] + [A-BF_iC_i]^TK + Q + \rho C_i^TF_i^TNF_iC_i = 0$
(solve for K)

(c)
$$\left.\begin{aligned} & L[A-BF_iC_i]^T + [A-BF_iC_i]L + X_o = 0 \\ & F_i = \frac{1}{\rho}N^{-1}B^TKLC_i^T[C_iLC_i^T]^{-1} \end{aligned}\right\} \text{solve iteratively for } F_i, L$$

YES

⋮

(d) Continue Levine-Athans-Johnson computations in this pass?

NO

(7) $q_i \geq q_{add}$? —— YES ⟶ OUT

NO ↓

YES CONTINUE COMPUTATIONS?

↓ NO

OUT

A tilde is used with L in step 5 above to indicate that this is not numerically the same matrix L that is computed in step 6. When using the above algorithm, it is likely to be convenient and computationally efficient to perform an a priori transformation on $\underline{x}$ to yield $C = [I_r \vdots 0]$. When adding rows to C to form C_o in step 1, one then can merely use rows which each contain all 0's except for a single 1 in the proper place to produce $C_o = I_n$. The Levine-Athans-Johnson algorithm shown in step 6 is the so-called Method 1 which was defined earlier when describing Söderström's paper; Söderström's results provide some indication that this approach is more stable than Method 2.

5.3 Application of Optimal Output Feedback to Simple Spring-Mass Problem

This section describes the results which were obtained when the optimal output feedback controller design technique was applied to the two degree of freedom spring-mass problem being considered with all the output feedback design methods discussed in this report.

The plant and its parameters and variables have been described in Section 2. The present work will consider only the case where actuators are mounted on both masses. The Levine-Athans-Johnson algorithm is, however, applicable also to the two cases where only one actuator is used.

The plant equations of motion in physical coordinates q are

$$M\ddot{\underline{q}} + K\underline{q} = \underline{u} \tag{5-13}$$

where

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad K = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \tag{5-14}$$

$m_1 = 1 \quad m_2 = 2 \quad k_1 = 1 \quad k_2 = 4$

and u_1, u_2 are the actuator forces

Transforming equations (5-13) and (5-14) to normal mode coordinates $\underline{\eta}$ yields

$$\ddot{\underline{\eta}} + \Omega^2\underline{\eta} = \Phi^T\underline{u}$$

$$\text{where } \Phi = [\underline{\phi}_1\underline{\phi}_2] = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} = \begin{bmatrix} .515499 & -.856890 \\ .605912 & .364512 \end{bmatrix} \tag{5-15}$$

$$\Omega^2 = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} = \begin{bmatrix} .5463^2 & 0 \\ 0 & 2.589^2 \end{bmatrix} \tag{5-16}$$

and the vectors $\underline{\eta}$ and $\underline{q}$ are related through

$$\underline{q} = \Phi\,\underline{\eta}$$

Transforming the modal equation into state vector form produces

$$\dot{\underline{x}}_T = A_T\underline{x}_T + B_T\underline{u}$$

where

$$\underline{x}_T = \{\dot{\eta}_1, \dot{\eta}_2, \eta_1, \eta_2\}^T$$

$$A_T = \begin{bmatrix} 0 & -\Omega^2 \\ I_2 & 0 \end{bmatrix}$$

$$B_T = \begin{bmatrix} \Phi^T \\ 0 \end{bmatrix}$$

The superscript T above denotes transposition and the subscript T denotes "total".

It is assumed that the plant possesses sensors which measure either $\dot{q}_1$ or $\dot{q}_2$. (Preliminary study showed that the Levine-Athans-Johnson algorithm cannot be applied if both $\dot{q}_1$ and $\dot{q}_2$ are sensed because then C does not have full rank, and the matrix CLC^T in equation (5-12) hence cannot be inverted.†) The equation for the sensor output y therefore is

$$y = \dot{q}_1 = \underline{c}_T^T\underline{x}_T$$

† New results reported in Section 6.2.3 may allow this restriction to be lifted.

where

$$\underline{c}_T^T = \{\phi_{i1}\phi_{i2}\ 0\ 0\}\underline{x}_T$$

The subscript i above equals either 1 or 2 to indicate which sensor is being employed.

The equation for the controller will be

$$\underline{u} = -\underline{f}y$$

As discussed in Section 2, the goals of this study are: (1) to design a controller assuming the lower frequency mode (mode 1) to be nonexistent, and then (2) to evaluate the performance of the overall system which includes mode 1, mode 2, and the controller.

When mode 1 is deleted, the pertinent equations on the preceding page become

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{5-17}$$

$$y = \underline{c}^T\underline{x} \tag{5-18}$$

$$\underline{u} = -\underline{f}y \tag{5-19}$$

where

$$\underline{x} = \{x_2, x_4\}^T = \{\dot{\eta}_2, \eta_2\}^T \tag{5-20}$$

$$A = \begin{bmatrix} 0 & -\omega_2^2 \\ 1 & 0 \end{bmatrix} \tag{5-21}$$

$$B = \begin{bmatrix} \underline{\phi}_2^T \\ \underline{0}^T \end{bmatrix} = \begin{bmatrix} \phi_{12} & \phi_{22} \\ 0 & 0 \end{bmatrix} \tag{5-22}$$

$$\underline{c}^T = \{\phi_{12}, 0\} \tag{5-23}$$

In the present application, the equations given earlier for the optimal output feedback algorithm can be written in the form

$$\tilde{A} = A - B\underline{f}\underline{c}^T \tag{5-24}$$

$$\tilde{Q} = Q + \rho\underline{c}\underline{f}^T N\underline{f}\underline{c}^T \tag{5-25}$$

$$\tilde{KA} + \tilde{A}^T K + \tilde{Q} = 0 \tag{5-26}$$

$$\tilde{A}L + L\tilde{A}^T + X_o = 0 \tag{5-27}$$

$$\underline{f} = \frac{1}{\rho}[\underline{c}^T L \underline{c}]^{-1} N^{-1} B^T K L \underline{c} \tag{5-28}$$

The above equations must be solved for the 2×1 gain vector $\underline{f}$. First, however, it is necessary to choose the elements of the 2×2 weighting matrices Q and N and of the 2×2 initial state covariance matrix X_o. The following were selected

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \tag{5-29}$$

$$N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5-30}$$

$$X_o = \beta \begin{bmatrix} 1 & 0 \\ 0 & \omega_2^{-2} \end{bmatrix} \tag{5-31}$$

Initially, the scalars β and ρ are carried along as unspecified parameters. The rationale for the choice of Q above is that it makes $\underline{x}^T Q \underline{x}$ proportional to the mechanical energy of the controlled mode. The selected matrix N causes the two control forces u_1 and u_2 to be weighted equally in the loss function J. The choice of X_o was based on the hypothesis that immediately prior to activating the regulator, η_2 is in undamped oscillation ($\eta_2(t) = \omega_2^{-1} \sqrt{2\beta} \sin \omega_2 t$) and that the regulator is actuated at a completely random point in this oscillation.

The equations which specify $\underline{f}$ are derived in the following paragraphs. Substituting equations (5-22), (5-23), and (5-30) into (5-28) produces

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{(k_{11} l_{11} + k_{12} l_{12})}{\rho l_{11} \phi_{12}} \begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix} \tag{5-32}$$

where l_{ij} and k_{ij} are the ij-th elements of L and K respectively. (The present work constrains K and L to be symmetric.) Equation (5-32) shows that f_1 and f_2 are related through

$$f_1 = \frac{\phi_{12}}{\phi_{22}} f_2 \tag{5-33}$$

Equation (5-33) provides a constraint between f_1 and f_2; it is an important result.

The next task is to determine an equation for f_2. Substituting equations (5-21) through (5-23) into (5-24) yields

$$\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & -\omega_2^2 \\ 1 & 0 \end{bmatrix} \tag{5-34}$$

where

$$\tilde{a}_{11} = -\phi_{12}[f_1\phi_{12} + f_2\phi_{22}] \tag{5-35}$$

Similarly, substituting equations (5-29), (5-23), and (5-30) into (5-25) produces

$$\tilde{Q} = \begin{bmatrix} \tilde{q}_{11} & 0 \\ 0 & \omega_2^2 \end{bmatrix} \tag{5-36}$$

where

$$\tilde{q}_{11} = 1 + \rho\phi_{12}^2[f_1^2 + f_2^2] \tag{5-37}$$

Subsequent work is restricted to the case where f_1 and f_2 are related as in equation (5-33). Then, equations (5-35) and (5-37) reduce to

$$\tilde{a}_{11} = -\frac{\phi_{12}}{\phi_{22}}[\phi_{12}^2 + \phi_{22}^2]f_2 \tag{5-35'}$$

$$\tilde{q}_{11} = 1 + \frac{\rho\phi_{12}^2[\phi_{12}^2 + \phi_{22}^2]f_2^2}{\phi_{22}^2} \tag{5-37'}$$

Substituting equations (5-34) and (5-36) into equation (5-26) yields a set of equations which can be solved for the elements of K. The results for k_{11} and k_{12} are

$$k_{11} = \frac{-[1 + \tilde{q}_{11}]}{2\tilde{a}_{11}} \tag{5-38}$$

$$k_{12} = \frac{1}{2} \tag{5-39}$$

Similarly, substituting equations (5-34) and (5-31) into (5-27) yields a set of equations which can be solved for the elements of L. The results for l_{11} and l_{12} are

$$l_{11} = -\frac{\beta}{\tilde{a}_{11}} \tag{5-40}$$

$$l_{12} = -\frac{\beta}{2\omega_2^2} \tag{5-41}$$

Substituting equations (5-38) through (5-41) into the bottom scalar equation in (5-32) yields

$$f_2 = -\frac{\phi_{22}[2\omega_2^2(1 + \tilde{q}_{11}) - \tilde{a}_{11}^2]}{4\phi_{i2}\rho\tilde{a}_{11}\omega_2^2} \tag{5-42}$$

Substituting equations (5-35') and (5-37') into (5-42) and rearranging produces

$$4\omega_2^2\phi_{22}^2 = f_2^2\phi_{i2}^2(\phi_{12}^2 + \phi_{22}^2)(2\rho\omega_2^2 + \phi_{12}^2 + \phi_{22}^2)$$

which can be solved for f_2

$$f_2 = \frac{\pm 2\omega_2\phi_{22}}{\phi_{i2}[(\phi_{12}^2 + \phi_{22}^2)(2\rho\omega_2^2 + \phi_{12}^2 + \phi_{22}^2)]^{1/2}} \tag{5-43}$$

Equations (5-33) and (5-43) are the basic equations for the controller gains f_1 and f_2. It has not yet been verified, however, that the regulator will be stable for either or both of the sign conditions in equation (5-43). The material in the following paragraph, as a sidelight, enables this question to be answered.

In equation (5-43), the gain f_2 is specified as a function of the control energy weighting gain ρ. Since ρ is actually of little intrinsic value, it would be convenient if ρ, or f_2 itself, could be expressed as a function of the damping ratio ζ_c of the regulator. This can be accomplished through use of the closed loop system characteristic equation

$$\det[sI_2 - \tilde{A}] = 0 \tag{5-44}$$

Substituting equation (5-34) into (5-44) yields

$$s^2 - \tilde{a}_{11}s + \omega_2^2 = 0$$

Hence

$$\zeta_c = \frac{-\tilde{a}_{11}}{2\omega_2} \tag{5-45}$$

Substituting equation (5-35') into (5-45) yields a relation between f_2 and ζ_c

$$f_2 = \frac{2\omega_2\phi_{22}\zeta_c}{\phi_{12}[\phi_{12}^2 + \phi_{22}^2]} \tag{5-46}$$

A relation between ζ_c and ρ can be obtained by inserting equation (5-43) into equation (5-46) and solving for ρ

$$\rho = \frac{(\phi_{12}^2 + \phi_{22}^2)}{2\omega_2^2}\left(\frac{1}{\zeta_c^2} - 1\right) \tag{5-47}$$

Use of equation (5-46) directly will produce the same value of f_2 as would be obtained by use of equations (5-47) and (5-43). The former procedure, however, tends to violate the spirit of the optimal output feedback design procedure.

Recall, from equation (5-15), that $\phi_{22} > 0$. Equation (5-46) thus shows that the regulator will be stable (at least for the one-mode plant being considered in the design) if and only if f_2 and ϕ_{12} have the same sign. This indicates that the positive sign in equation (5-43) must be chosen for both $i = 1$ and $i = 2$.

In summary, the final equations for the regulator design are equations (5-47), (5-43), and (5-33). The numerical values of the plant parameters are listed in equations (5-15) and (5-16). The value of ζ_c must be chosen a priori. Selecting the ζ_c of 0.1 which was specified in the problem statement yields the results shown in Table 5-1.

Table 5-1: Optimal output feedback gains for sample problem.

i	ρ	f_1	f_2
1	6.404	0.597	-0.254
2	6.404	-1.404	0.597

The reader is reminded that $i = 1$ signifies the system in which the velocity sensor is mounted on mass 1, while $i = 2$ signifies the system in which this sensor is mounted on mass 2.

The remaining work in this section consists of an investigation of the performance of the two control systems specified above on the full two degree of freedom plant. That is, the effect of the residual mode (mode 1) which was neglected when designing the control system will be studied. In particular, we wish to address the same questions which are being addressed in the other sections of this report where the other output feedback controller design techniques are applied to the sample problem.

The first task is to determine whether or not the residual mode destabilizes either or both of the control systems indicated on Table 5-1. For generality, the approach which will be used consists of: (1) determining the regions of stability and instability in an $f_1 - f_2$ gain space (where f_1 and f_2 are independent), and (2) checking in which region the regulators in Table 5-1 lie.

For the immediate purposes, it will be convenient to employ the physical variables q. Consider first the case (i = 1) where the sensor is mounted on mass 1. Then

$$\underline{u} = -\underline{f}\dot{q}_1 \qquad (5\text{-}48)$$

Inserting equation (5-48) into (5-13) and taking the Laplace transform yields

$$\begin{bmatrix} m_1 s^2 + f_1 s + k_1 + k_2 & -k_2 \\ f_2 s - k_2 & m_2 s^2 + k_2 \end{bmatrix} \begin{Bmatrix} q_1(s) \\ q_2(s) \end{Bmatrix} = \underline{h}(s)$$

where $\underline{h}(s)$ is a function of the initial conditions. The characteristic equation is the determinant of the matrix on the left, set to 0. Expanding this determinant yields

$$m_1 m_2 s^4 + m_2 f_1 s^3 + [m_1 k_2 + m_2(k_1 + k_2)]s^2 + k_2(f_1 + f_2)s + k_1 k_2 = 0 \qquad (5\text{-}49)$$

Substituting the values of m_1, m_2, k_1, and k_2 into equation (5-49) produces

$$s^4 + f_1 s^3 + 7s^2 + 2(f_1 + f_2)s + 2 = 0 \qquad (5\text{-}50)$$

When Routh's criteria is applied to equation (5-50), the terms in the first column turn out to be

$$1, \quad f_1, \quad \frac{1}{f_1}[5f_1 - 2f_2], \quad \frac{-4[f_2 + 0.8508f_1][f_2 - 2.3508f_1]}{(5f_1 - 2f_2)}, \quad 2$$

from which necessary and sufficient conditions for stability for i = 1 can be determined to be

$$f_1 > 0$$

$$-0.8508\, f_1 < f_2 < 2.3508\, f_1$$

Similarly, when the sensor is on mass 2 (i = 2) we have

$$\underline{u} = \underline{f}\dot{q}_2$$

which yields the characteristic equation

$$\det \begin{bmatrix} m_1 s^2 + k_1 + k_2 & f_1 s - k_2 \\ -k_2 & m_2 s^2 + f_2 s + k_2 \end{bmatrix}$$

$$= m_1 m_2 s^4 + m_1 f_2 s^3 + [m_1 k_2 + m_2(k_1 + k_2)] s^2$$

$$+ s[f_2(k_1 + k_2) + f_1 k_2] + k_1 k_2 = 0 \qquad (5\text{-}51)$$

Substituting numerical values into equation (5-51) provides

$$s^4 + 0.5 f_2 s^3 + 7s^2 + [2f_1 + 2.5f_2] s + 2 = 0 \qquad (5\text{-}52)$$

Applying Routh's criteria to equation (5-52) yields the following terms in column 1

$$1,\ 0.5f_2,\ \frac{2}{f_2}[f_2 - 2f_1],\ \frac{2}{(f_2 - 2f_1)}[f_2 + 0.8508 f_1][f_2 - 2.3508 f_1],\ 2$$

from which necessary and sufficient conditions for stability for i = 2 can be determined to be

$$f_2 > 0$$

$$f_2 > 2.3508\, f_1$$

$$f_2 < -0.8508\, f_1$$

The $f_1 - f_2$ stability regions determined above for cases i = 1 and i = 2 are plotted on Figure 5-1. The figure also shows the optimal output feedback controllers indicated previously in Table 5-1. It is seen that the optimal gains yield a stable system for the i = 1 case and an unstable one for the i = 2

case. The instability for $i = 2$ is due to the residual mode which was omitted when designing the controller.

The derivation given earlier in this section indicated that the control weighting matrix ($N = I_2$) chosen for this problem yielded the constraint $f_2 = (\phi_{12}/\phi_{22})f_2$ between f_1 and f_2. A plot of this line is included on Figure 5-1. The sole effect of ρ, X_0, and Q is to determine the precise point on this line where the optimal gains f_1 and f_2 lie. It is concluded from Figure 5-1 and equation (5-43) that with the selected X_0 and Q there is no choice of $\rho > 0$ that would make the $i = 1$ case unstable nor the $i = 2$ case stable. The negative sign for $i = 2$ shows immediately that the system is unstable as predicted by the preceding Routh's criteria analysis.

Substituting the f_1 and f_2 values from Table 5-1 into equations (5-50) and (5-52) produces the following characteristic equations for the optimal regulator:

$$i = 1: \quad s^4 + 0.597s^3 + 7s^2 + 0.686s + 2 = 0$$

$$i = 2: \quad s^4 + 0.2985s^3 + 7s^2 - 1.3155s + 2 = 0$$

The damping factors ζ and natural frequencies ω of the regulator can be obtained by factoring the above expressions. The results are

i	ω_1	ζ_1	ω_2	ζ_2
1	0.5481	0.07805	2.5804	0.1002
2	0.5417	-0.1985	2.6109	0.09836

Comparison of the ω's above with the open loop plant values indicated earlier shows that the controller has produced almost no effect on the system natural frequencies. The proximity of the ζ_2 results shown above to the design goal of $\zeta_2 = 0.1$ is surprising, since mode 1 was completely disregarded in the design.

The remainder of this section will not consider further the unstable system, $i = 2$. Instead, the work henceforth will be limited to the stable case $i = 1$, in which the velocity sensor is mounted on mass 1.

The next portion of this section will be devoted to setting up transfer functions for the purpose of determining the frequency response and transient response requested in the problem statement. To obtain these, we first combine equations (5-13), (5-14), and (5-48) into the form:

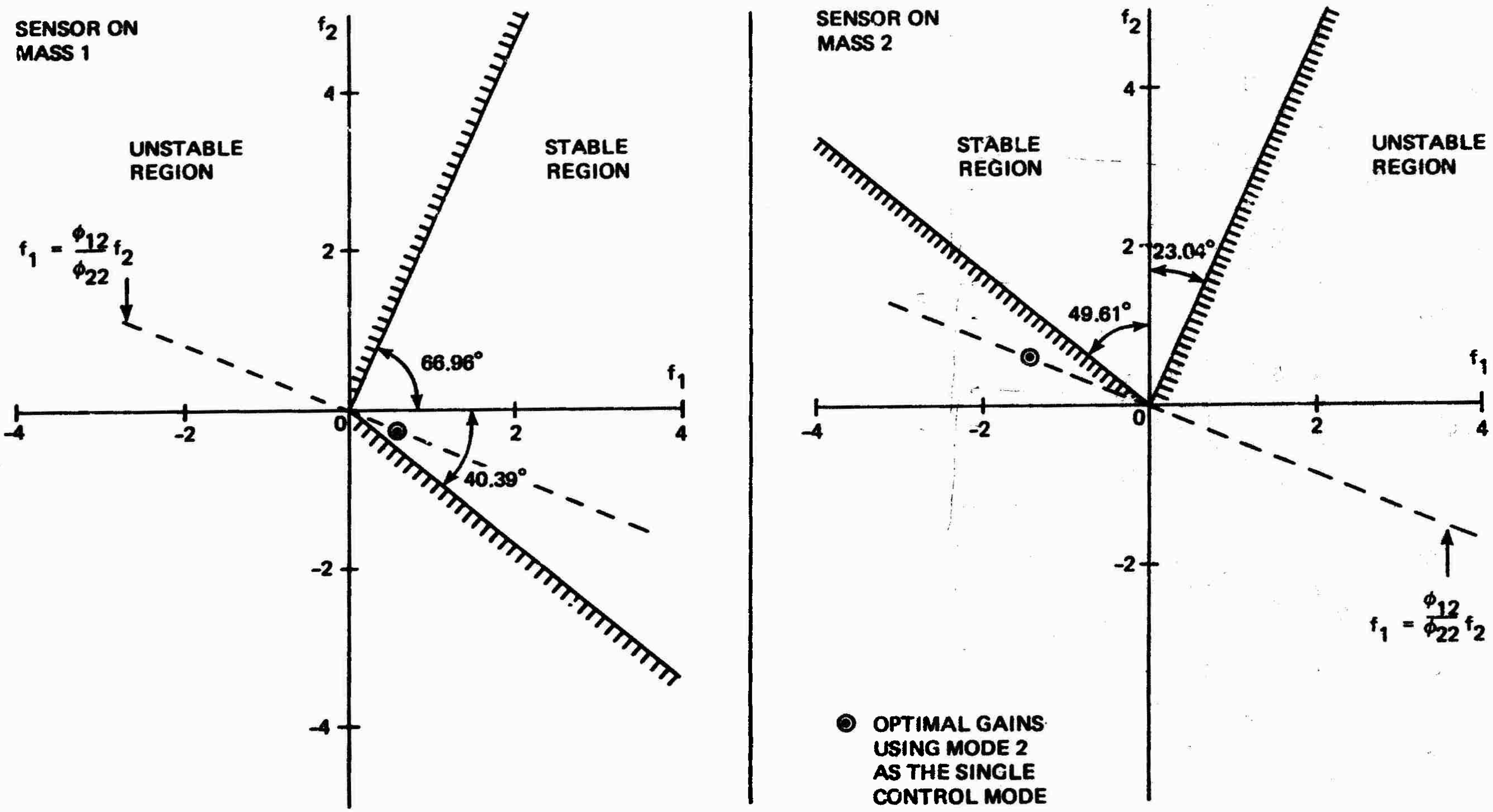


Figure 5-1. Stability-instability regions for sample problem.

$$m_1\ddot{q}_1 = v_1 - f_1\dot{q}_1 - (k_1 + k_2)q_1 + k_2q_2$$

$$m_2\ddot{q}_2 = v_2 - f_2\dot{q}_1 + k_2q_1 - k_2q_2$$

The terms v_1 and v_2 above are disturbance forces which have been added to the model. The equations assume that the velocity sensor is mounted on mass 1. However, they do not assume that the controller gains f_1 and f_2 are established optimally. The signal flow diagram shown as Figure 5-2 can be obtained easily from the equations. Transfer functions between any desired input and output variables on the diagram can be obtained by use of Mason's rule. The two transfer functions which are of current interest are

$$\frac{q_2(s)}{v_2(s)} = \frac{m_1s^2 + f_1s + (k_1 + k_2)}{\text{Den}}$$

$$\frac{q_2(s)}{q_2(0)} = \frac{m_2s[m_1s^2 + f_1s + (k_1 + k_2)]}{\text{Den}}$$

where

$$\text{Den} = m_1m_2s^4 + m_2f_1s^3 + [m_2(k_1 + k_2) + m_1k_2]s^2 + k_2(f_1 + f_2)s + k_1k_2$$

Note that $q_2(s)$ above is the Laplace transformed variable, while $q_2(0)$ is the initial condition. The denominator Den is identical to the characteristic equation result given earlier as equation (5-49).

Substituting the numerical values for the m's and k's into the above expressions yields

$$\frac{q_2(s)}{v_2(s)} = \frac{0.5[s^2 + f_1s + 5]}{\text{Den}}$$

$$\frac{q_2(s)}{q_2(0)} = \frac{s[s^2 + f_1s + 5]}{\text{Den}}$$

where

$$\text{Den} = s^4 + f_1s^3 + 7s^2 + 2[f_1 + f_2]s + 2$$

If f_1 and f_2 are constrained by the relation $f_1 = -2.3508\ f_2$, which was an intermediate result when deriving the optimal output feedback controller, the above expressions become

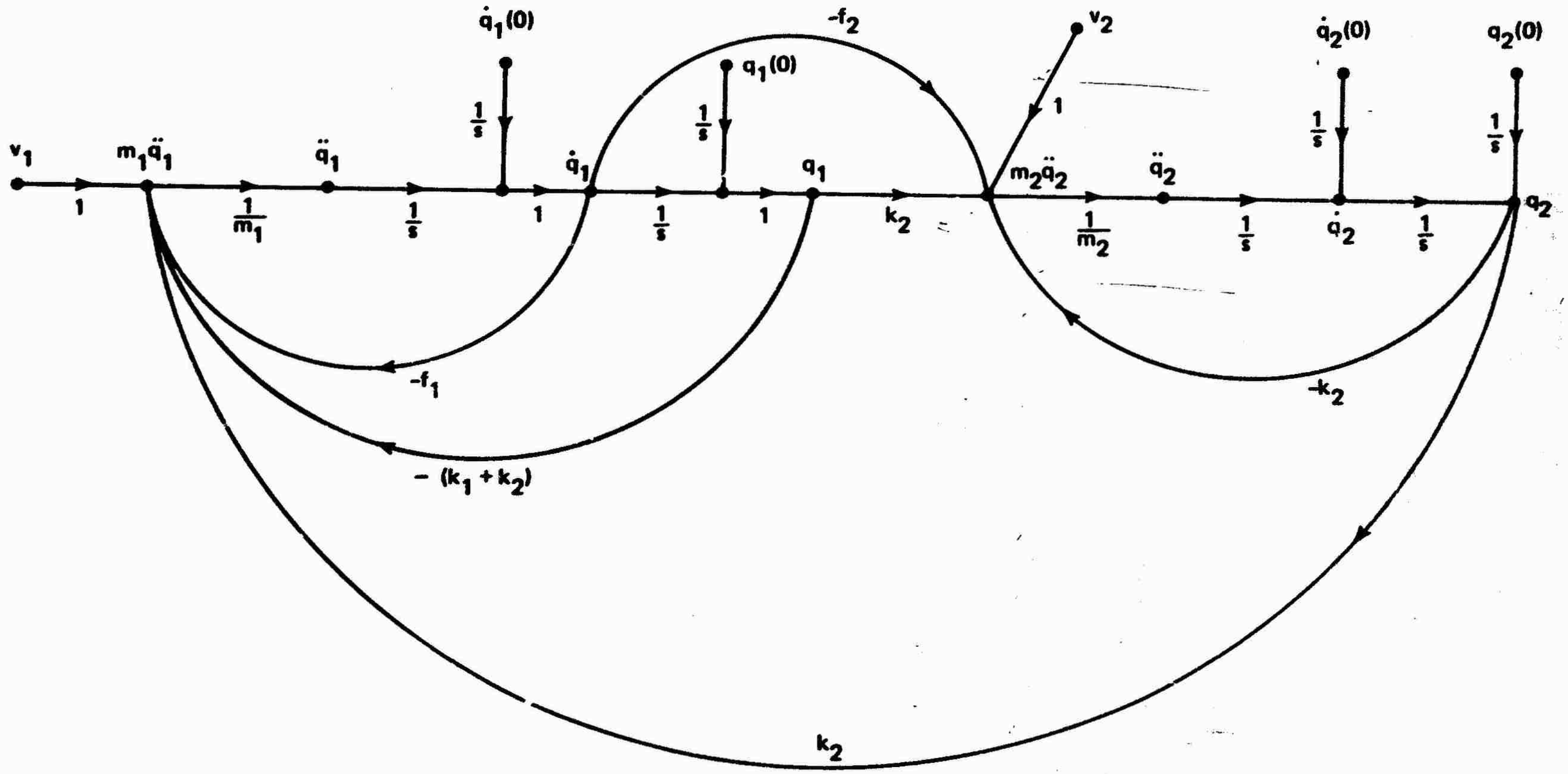


Figure 5-2. Signal flow diagram for sample problem.

$$\frac{q_2(s)}{v_2(s)} = \frac{0.5[s^2 - 2.3508f_2s + 5]}{Den}$$

$$\frac{q_2(s)}{q_2(0)} = \frac{s[s^2 - 2.3508f_2s + 5]}{Den}$$

where

$$Den = s^4 - 2.3508f_2s^3 + 7s^2 - 2.7016f_2s + 2$$

The final transfer functions for the optimal controller for $i = 1$ are obtained by substituting $f_2 = -0.254$ from Table 5-1 into the above equations. After factoring Den, the result is

$$\frac{q_2(s)}{v_2(s)} = \frac{0.5[s^2 + 2 \times 0.1335 \times 2.236s + 2.236^2]}{Den}$$

$$\frac{q_2(s)}{q_2(0)} = \frac{s[s^2 + 2 \times 0.1335 \times 2.236s + 2.236^2]}{Den} \tag{5-53}$$

$$Den = (s^2 + 2 \times 0.1002 \times 2.5804s + 2.5804^2)$$
$$\cdot(s^2 + 2 \times 0.07805 \times 0.5481s + 0.5481^2)$$

Figure 5-3 is a frequency response plot of q_2/v_2. This plot was obtained using the transfer functions developed above. Figure 5-4 gives the response of the uncontrolled plant ($f_1 = f_2 = 0$). The work shows that at $\omega = 3$ there is negligible difference in the q_2 amplitude for the two conditions. See also Figure 5-5, which shows the transient response to the $\omega = 3$ input.

The next topic to be considered is the time response of $q_2(t)$ due to an initial condition $q_1(0) = 0$, $q_2(0) = 1.0$ when the optimal output feedback regulator is used. One way to proceed is by taking the inverse Laplace transform of Eq. (5-53). The result is

$$\frac{q_2(t)}{q_2(0)} = 0.7408e^{-\zeta_1\omega_1 t}\cos[\sqrt{1-\zeta_1^2}\,\omega_1 t + 4.175°]$$
$$+ 0.26470e^{-\zeta_2\omega_2 t}\cos[\sqrt{1-\zeta_2^2}\,\omega_2 t + 8.979°] \tag{5-54}$$

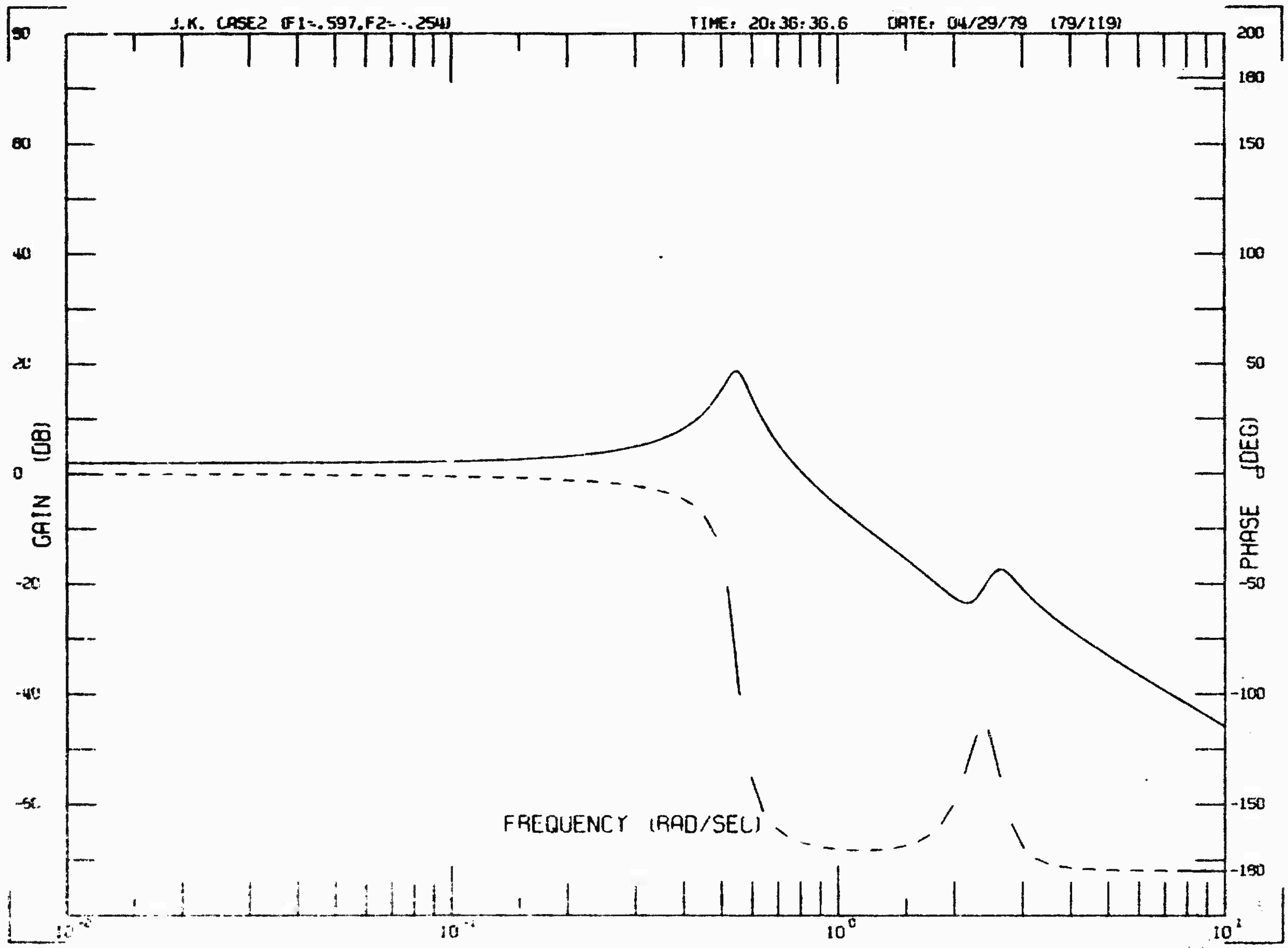


Figure 5-3. Closed loop frequency response for sample problem.

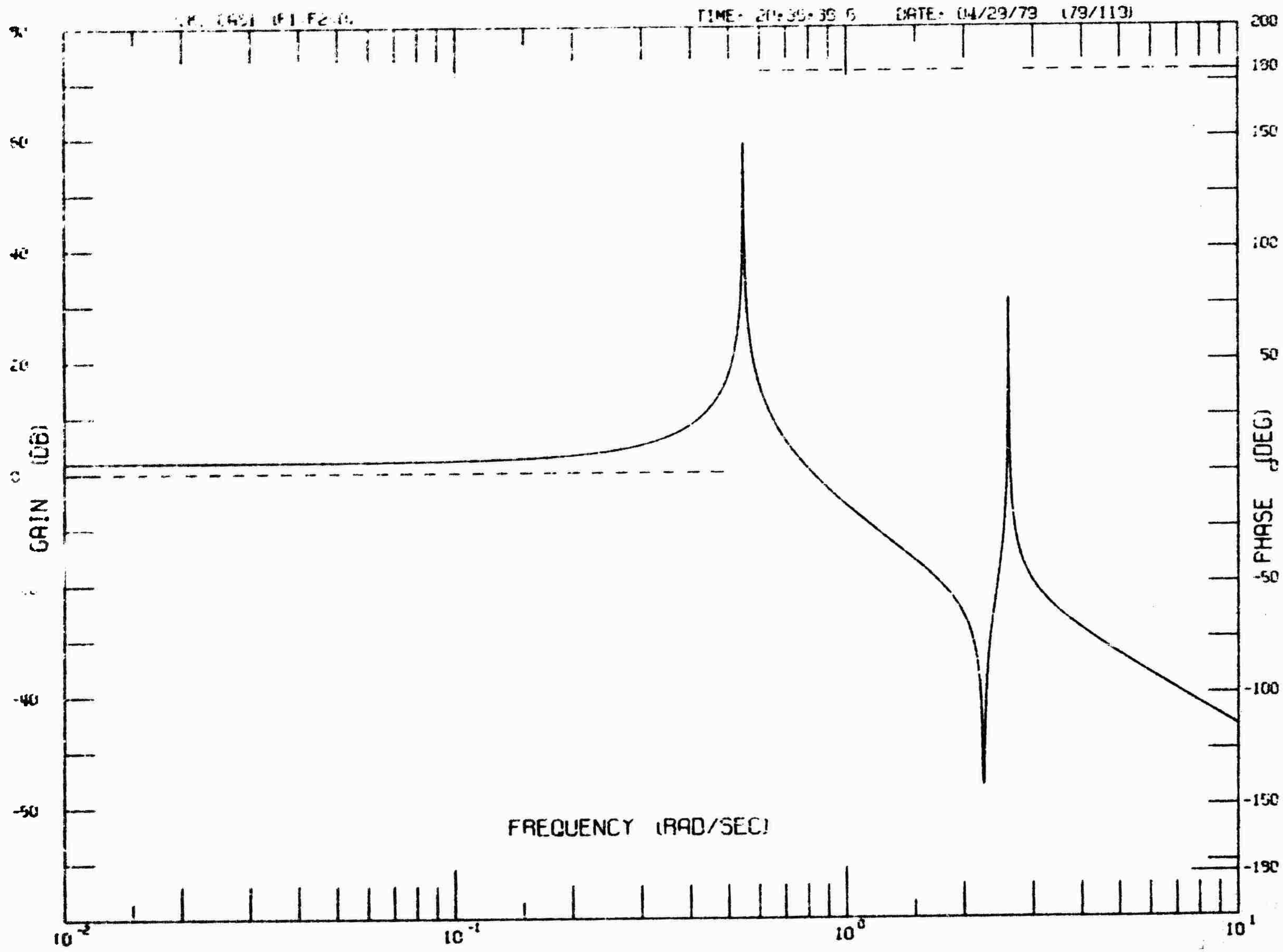


Figure 5-4. Open loop frequency response for sample problem.

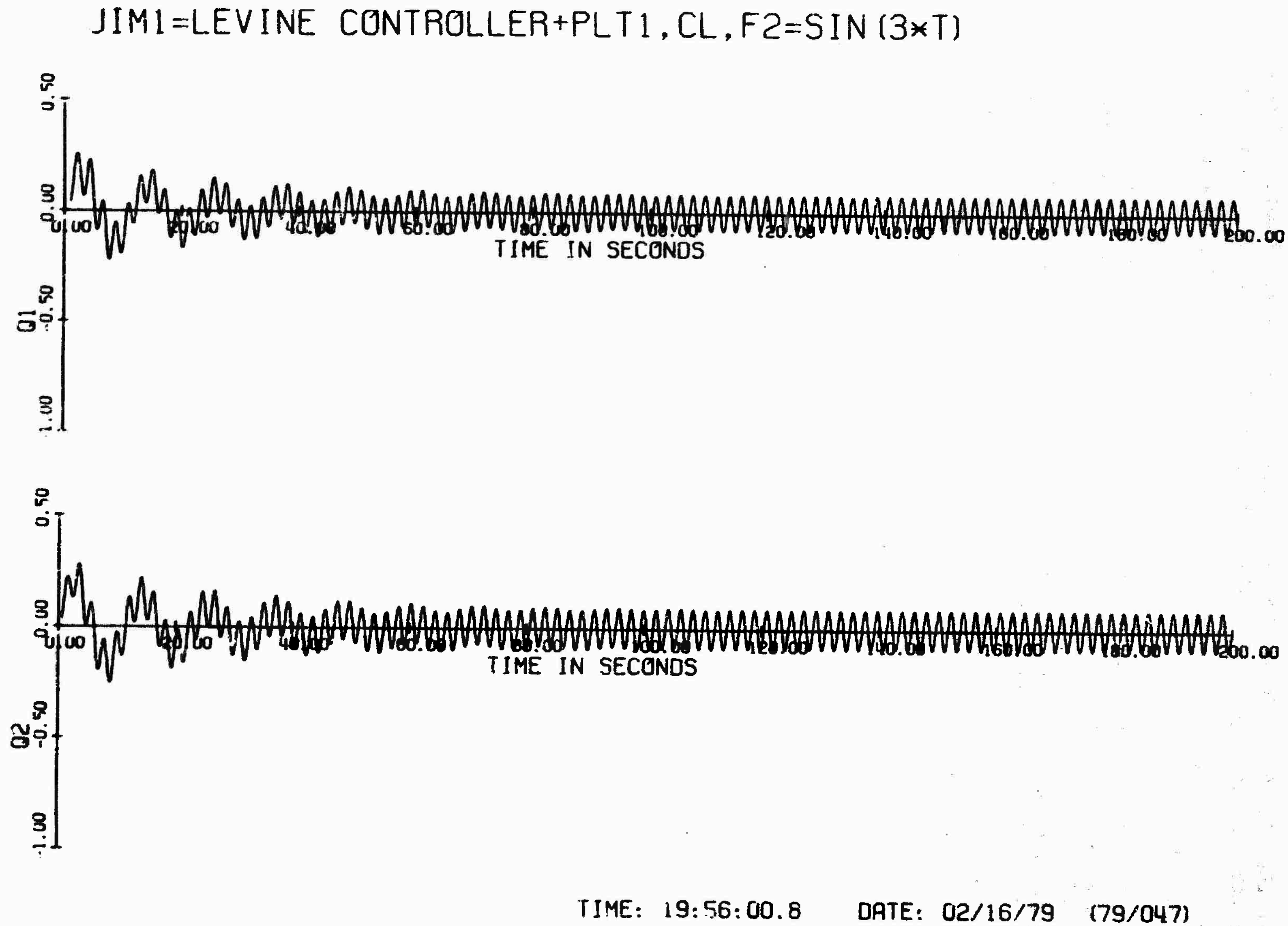


Figure 5-5. Response to periodic disturbance $f_2(t) = \sin 3t$.

where the ζ's and ω's have the values indicated earlier. Equation (5-54) is plotted as in Figure 5-6. The peak magnitude of $q_2(t)$ is seen to occur at $t = 0$. The steady state error is zero. The 5% settling time is approximately 59 seconds.

The final topics to be investigated in this section concern system robustness and the variation of the closed loop poles as ρ is varied throughout its full range: $\rho\varepsilon(0,\infty)$. The work still will be limited to the case $i = 1$. The gains f_1 and f_2 still will be constrained by equation (5-33). For the present work, it is slightly more convenient to employ f_1 rather than f_2. Equation (5-43) and the numerical data given earlier indicate that (when the optimal output feedback design technique is used and X_0, N, and Q are selected as indicated in equations (5-29) to (5-31)) the upper and lower limits of f_1 are

$$f_{1MIN} = 0 \quad (\rho \to \infty)$$

$$f_{1MAX} = \frac{2\omega_2}{\phi_{12}^2 + \phi_{22}^2} = 5.97 \quad (\rho \to 0)$$

The application of equation (5-33) to eliminate either f_1 or f_2 from the mathematics is useful because the system then can be studied using standard classical techniques (open loop frequency response plots or root locus plots with the sole design variable being a single gain). After equation (5-33) is used to eliminate f_2, the basic signal flow diagram given earlier as Figure 5-2 can be rearranged into the standard unity negative feedback one shown as Figure 5-7. This figure also includes an analogous signal flow diagram for the one-mode plant model which was used in designing the regulator. This second diagram can be obtained through use of equations (5-17) through (5-23) and (5-33).

The open loop transfer functions c/e can be obtained readily from Figure 5-7. They are

One Mode Plant Model

$$\frac{c}{e} = \frac{f_1 s[\phi_{12}^2 + \phi_{22}^2]}{s^2 + \omega_2^2}$$

$$= \frac{0.86713\ f_1 s}{(s + i2.589)(s - i2.589)} \qquad (5\text{-}55)$$

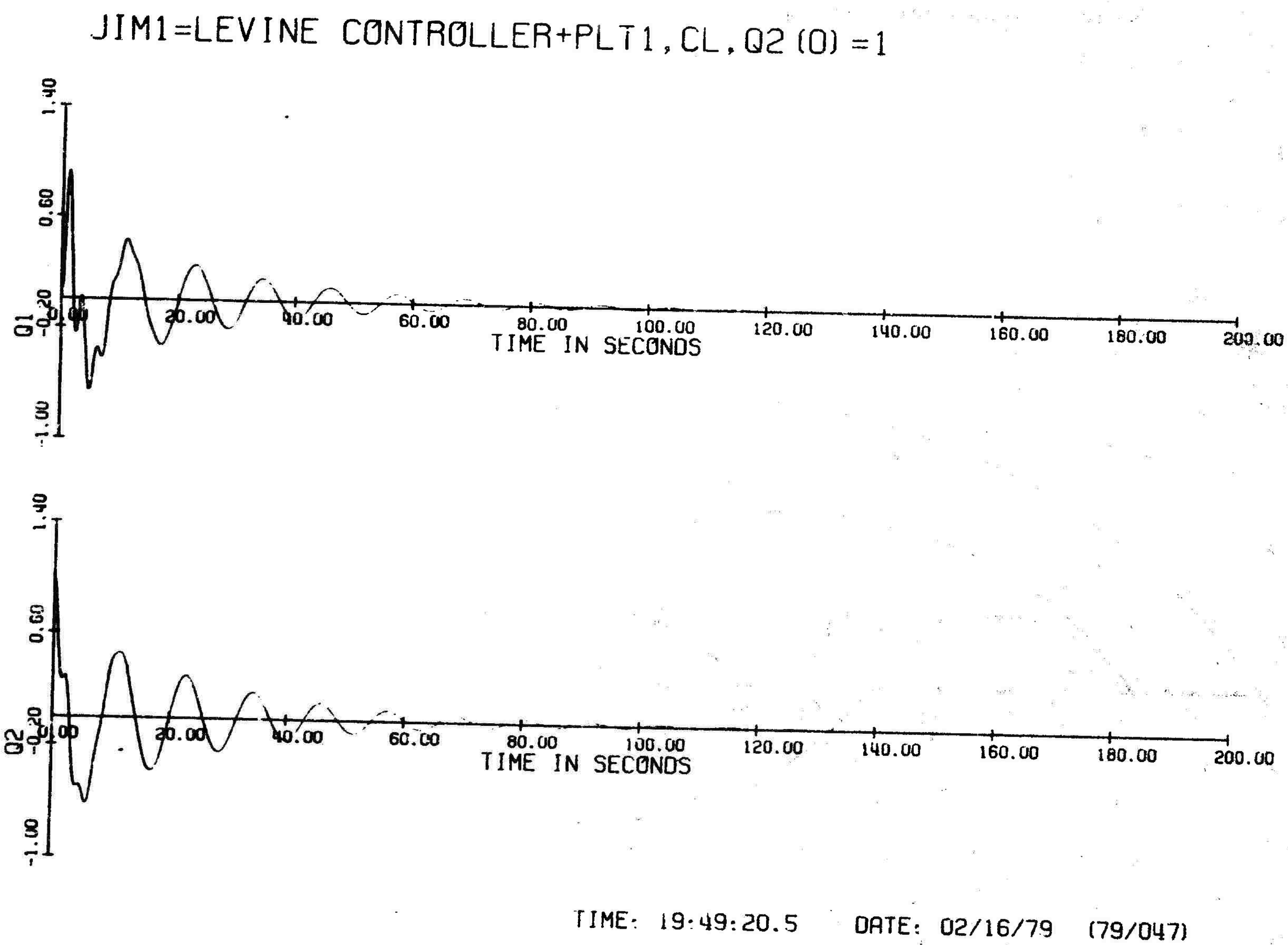


Figure 5-6. Response to initial conditions $q_2(0) = 1$.

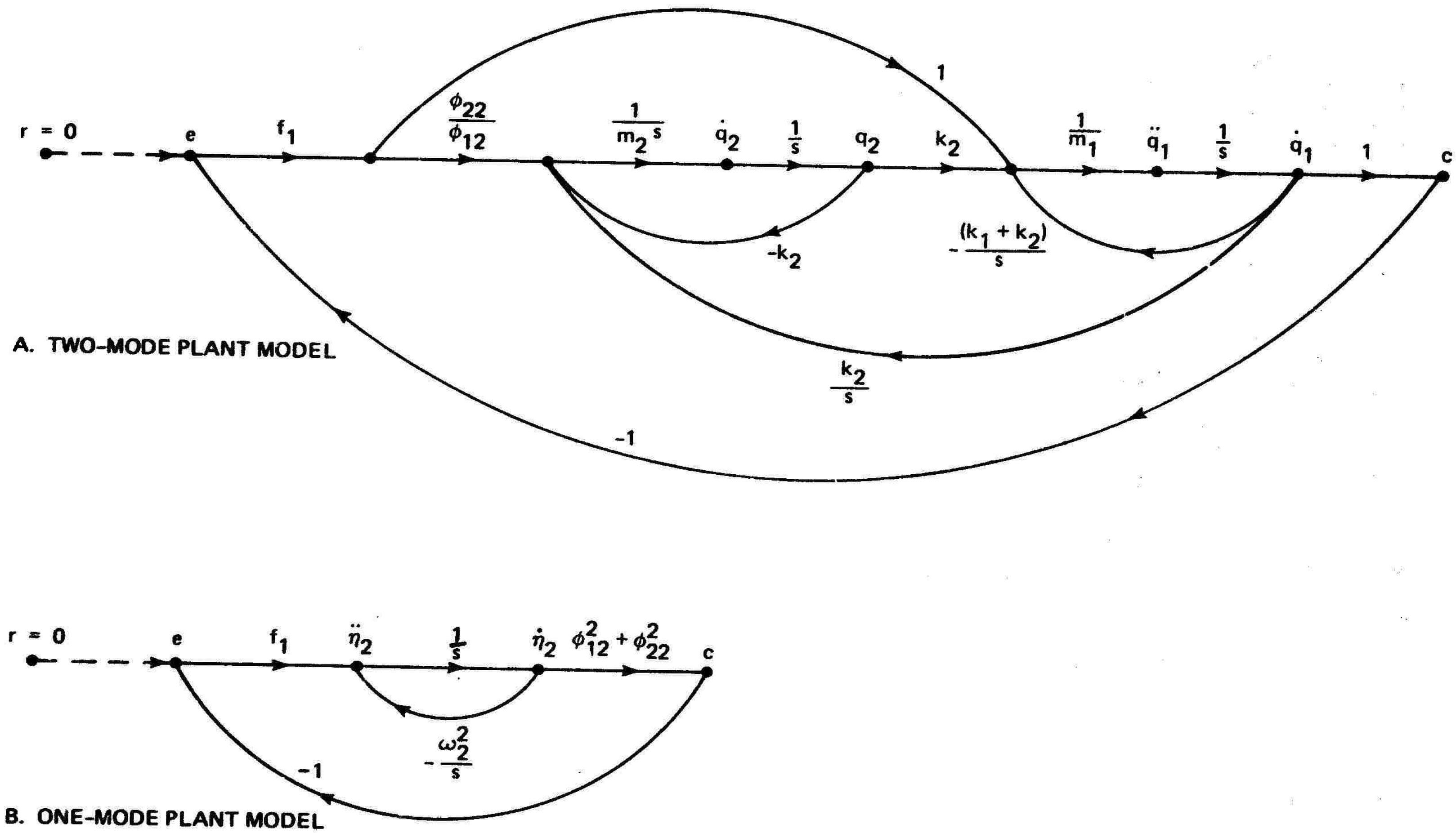


Figure 5-7. Alternate signal flow diagrams for sample problem.

Two Mode Plant Model

$$\frac{c}{e} = \frac{f_1 s\left[s^2 m_2 + k_2\left(1 + \frac{\phi_{22}}{\phi_{12}}\right)\right]}{s^4 m_1 m_2 + s^2 [m_1 k_2 + m_2(k_1 + k_2)] + k_1 k_2}$$

$$= \frac{f_1 s [s^2 + 1.1492]}{s^4 + 7s^2 + 2} \qquad (5\text{-}56)$$

$$= \frac{f_1 s [s + i1.0720][s - i1.0720]}{(s + i2.589)(s - i2.589)(s + i0.5463)(s - i0.5463)}$$

The poles in the above expressions are the natural frequencies of the plant.

Some quick frequency response sketches of equations (5-55) and (5-56) indicate immediately that in both cases the system possesses 90° phase margin and infinite gain margin. This is an overly-optimistic conclusion. It is a result of the fact that the plant models are idealized ones which do not include actuator or sensor lags.

Root locus plots of equations (5-55) and (5-56) are of some interest. Figure 5-8, which can be obtained easily from equation (5-55), shows the result for the one-mode plant model upon which the controller design was based. This plot indicates that the controller does not alter the natural frequency ω_2. It also indicates that (with the selected matrices Q and N) the upper limit on ζ which is achievable with optimal output feedback is unity.

Figure 5-9 is the root locus plot which was obtained from equation (5-56). The figure is in concurrence with previous conclusions that a stable system is obtained, for all values of ρ, even in the presence of this residual mode. As f_1 is increased, the pole-pair of this residual mode travels from the open loop values toward the open loop zero-pair of c/e. The pole-pair of the critical mode follows the same basic type of locus as in the one-mode condition shown previously in Figure 5-8. However, there are some differences. In particular, the breakaway point on the real axis is significantly different in the two cases. Also, on Figure 5-8 the upper limit of f_1(5.97) was just sufficient to drive to locus to the real-axis breakaway point. On Figure 5-9 an f_1 of 5.97 is seen to carry the system considerably beyond this point. Figure 5-9 indicates that the maximum achievable damping coefficient of the residual mode is approximately $\zeta = 0.59$; this is achieved at the real-axis breakaway point of the critical mode.

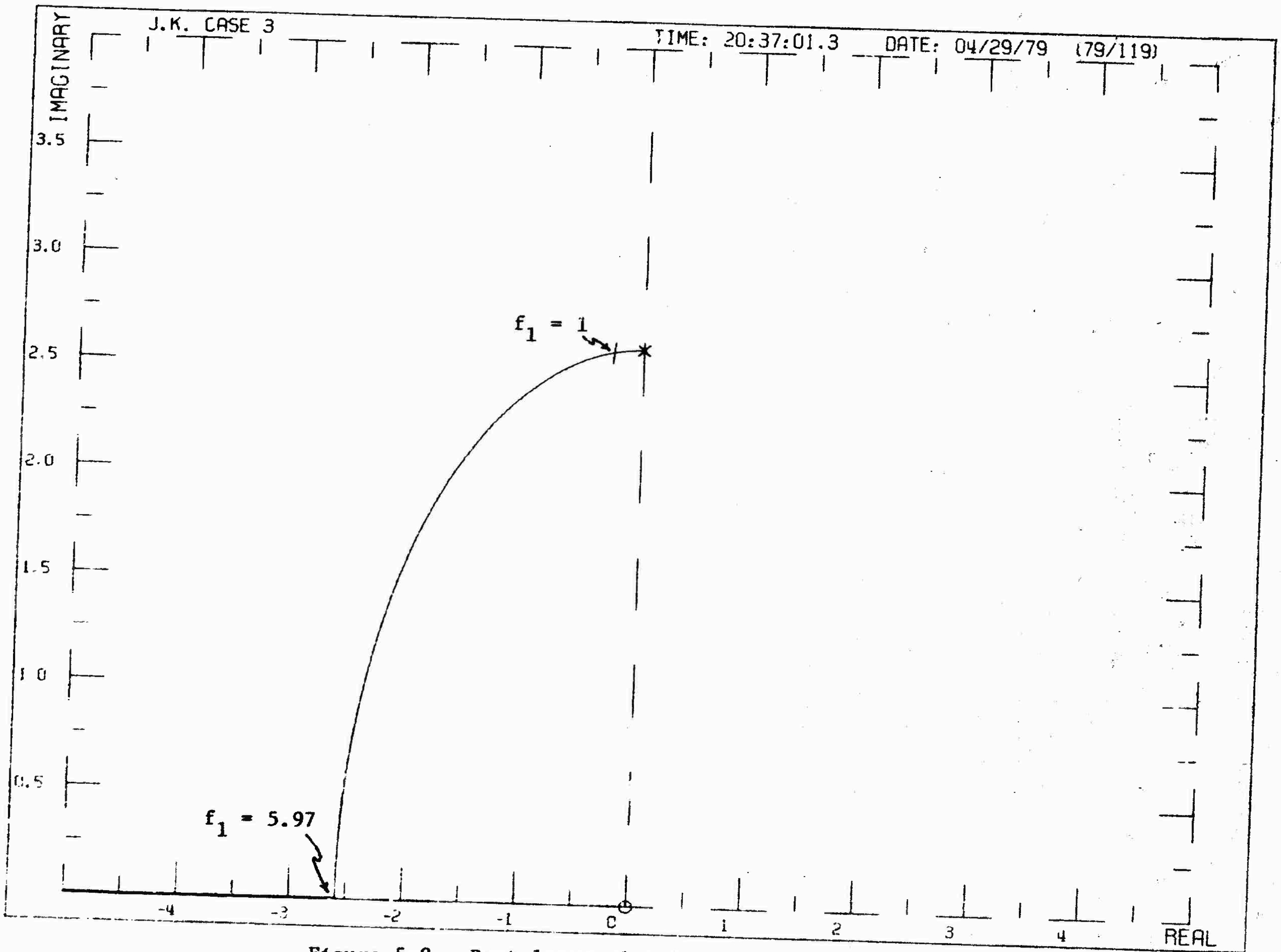


Figure 5-8. Root locus plot for one-mode plant model.

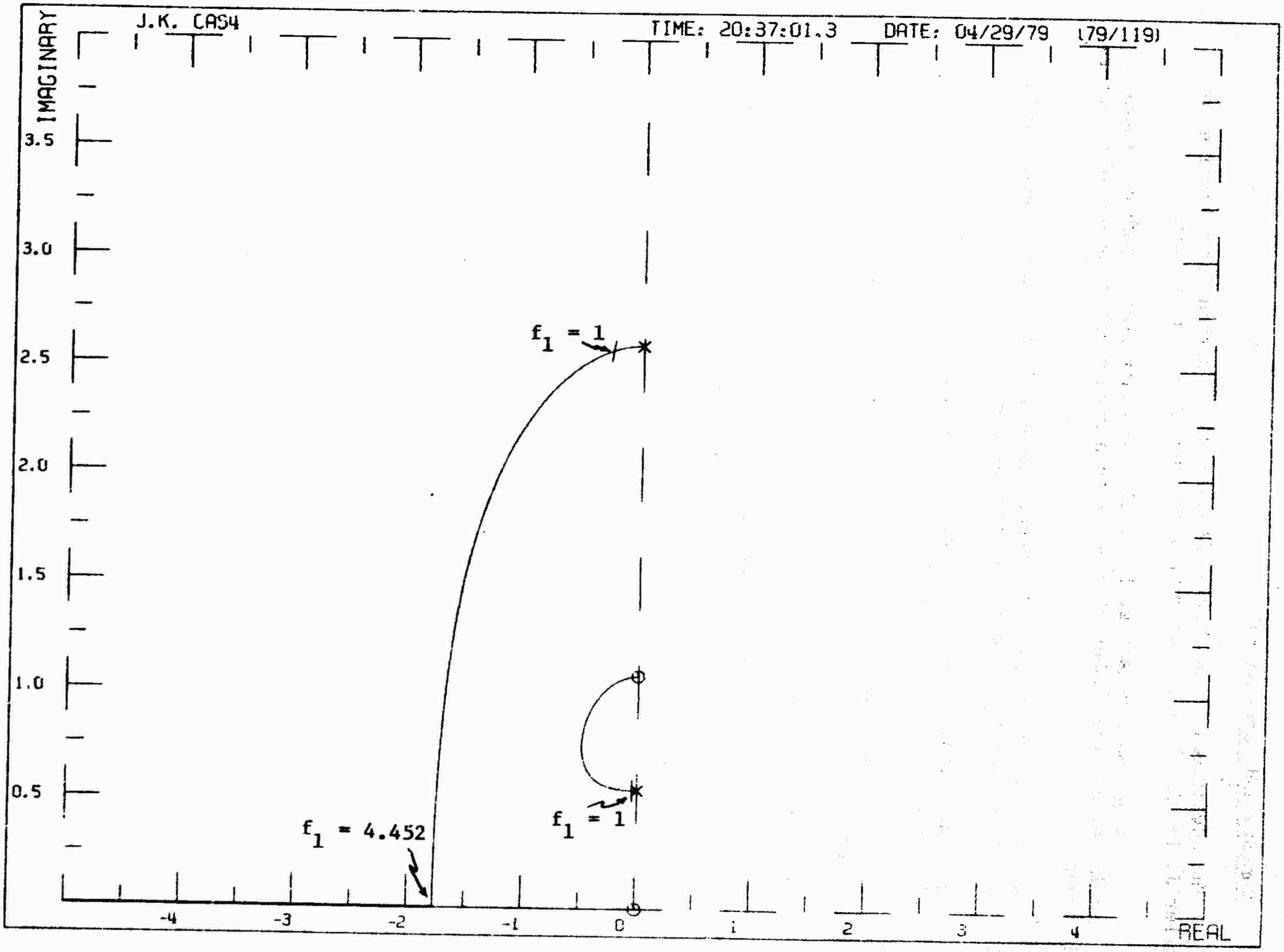


Figure 5-9. Root locus plot for two-mode plant model.

5.4 Summary and Conclusions

The advantages of the Levine-Athans-Johnson method for vibration controller design in the LSS application are summarized as follows.

1. It has some potential for designing controllers which can improve the damping of a large number of modes with a much smaller number of actuators and sensors.

2. A wide variety of performance characteristics can be achieved by selection of the weighting terms in the cost function J.

3. The technique is fairly well understood, since it has been the subject of numerous studies since its inception.

4. The technique yields a controller which, at least in a mathematical sense, is optimal.

The weaknesses are summarized as follows.

1. Determining optimal gains in the LSS applications generally is a very difficult problem. The main difficulties are:

   (a) the size of the matrices which are involved

   (b) convergence

   (c) local minima

Successful results depend on use of a very accurate initial estimate of the gain matrix F. The difficulty in obtaining a successful result increases as the ratio (number of modes)/(number of outputs) increases.

2. None of the studies reported in the literature considered problems nearly as large as those which normally will be encountered in LSS applications.

3. Controllers designed by the approach apparently have no known, guaranteed, robustness properties. The simple design example indicated that spillover has the potential for making such controllers unstable.

4. The Levine-Athans-Johnson algorithm cannot include constraints on the elements of F. Such constraints can, however, be incorporated if mathematical programming solution techniques are used.

Some final comments are

1. If the technique is implemented in a computer program, it should include at least one mathematical programming algorithm in place of, or in addition to, the solution-technique developed by Levine, Athans, and Johnson.

2. An implementation which performs successfully evidently will require some specific technique, such as that outlined earlier in this section, to provide an adequate initial estimate of F.

3. If the technique is implemented, considerable effort should be expended to find an adequate algorithm for solving the Liapunov-type equations.

References

1. Levine, W.S., and M. Athans, "On the Determination of the Optimal Constant Output Feedback Gains for Linear Multivariable Systems", IEEE Trans. Automatic Control, Vol. AC-15, pp. 44-48, February 1970.

2. Levine, W.S., T.L. Johnson, and M. Athans, "Optimal Limited State Variable Feedback Controls for Linear Systems", IEEE Trans. Automatic Control, Vol. AC-16, pp. 785-793, December 1971.

3. Kwakernaak, H., and R. Sivan, Linear Optimal Control Systems, Wiley-Interscience, New York, 1972.

4. Li, M.T., "On the Output Feedback Stabilizability of Linear Systems", IEEE Trans. Automatic Control, Vol. AC-17, pp. 408-410, June 1972.

5. Denham, M.J., "Stabilization of Linear Multivariable Systems by Output Feedback", IEEE Trans. Automatic Control, Vol. AC-18, pp. 62-63, February 1973.

6. Anderson, B.D.O., and J.B. Moore, Linear Optimal Control, Prentice-Hall, Englewood Cliffs, New Jersey, 1971.

7. Knapp, C.H., and S. Basuthakur, "On Optimal Output Feedback", IEEE Trans. Automatic Control, Vol. AC-17, pp. 823-825, December 1972.

8. Choi, S.S., and J.R. Sirisena, "Computation of Optimal Output Feedback Gains for Linear Multivariable Systems", IEEE Trans. Automatic Control, Vol. AC-19, pp. 257-258, June 1974.

9. Bingulac, S.P., N.M. Cuk, and M.S. Calovic, "Calculation of Optimum Feedback Gains for Output-Constrained Regulators", IEEE Trans. Automatic Control, Vol. AC-20, pp. 164-166, February 1975.

10. Petkovski, DJ.B., and M. Rakic, "On the Calculation of Optimum Feedback Gains for Output-Constrained Regulators", IEEE Trans. Automatic Control, Vol. AC-23, p. 760, August 1978.

11. Kosut, R.L., "Suboptimal Control of Linear Time-Invariant Systems Subject to Control Structure Constraints", IEEE Trans. Automatic Control, Vol. AC-15, pp. 557-563, October 1970.

12. Söderström, T., "On Some Algorithms for Design of Optimal Constrained Regulators", IEEE Trans. Automatic Control, Vol. AC-23, pp. 1100-1101, December 1978.

13. Knox, J.R., and J.M. McCarty, "Algorithms for Computation of Optimal Constrained Output Feedback for Linear Multivariable Flight Control Systems", AIAA paper 78-1290.

14. Smith, P., "Numerical Solutions of the Matrix Equation $AX + XA^T + B = 0$", IEEE Trans. Automatic Control, Vol. AC-16, pp. 278-279, June 1971.

SECTION 6

SUBOPTIMAL OUTPUT FEEDBACK CONTROL
VIA KOSUT APPROXIMATION

6.1 Background

6.1.1 Overview

One of the principal results that is known about the optimal regulation of linear dynamic systems with quadratic cost criteria [1] is that the optimal closed loop control is expressible as a linear function of the system state vector. No a priori constraints on the control vector are imposed in the development of this result. To implement this feedback control law in an actual control system would require either: measurements of the full system state, which is seldom possible; deterministic state reconstruction (Luenberger observer [2]); or stochastic state estimation (Kalman filter [3]). A desire to avoid the need for state reconstruction or estimation with systems in which full state measurements are not available has motivated studies of optimal output feedback [4], which constrains the feedback control law to be a linear function of the available measurements. Section 5 discussed such methods in detail. With such an a priori control structure constraint, the variational problem may be reduced to a parameter optimization problem. Necessary conditions consist of a system of coupled nonlinear algebraic equations for a cost matrix, an adjoint multiplier matrix, and a feedback gain matrix. Unfortunately, these algebraic necessary conditions cannot in general be solved in closed form; moreover, serious convergence difficulties have plagued attempts to develop algorithms for solution by iteration [4], [5]. A principal objective of the Kosut approach to the output feedback problem [6] is to develop design methods which avoid the need for an iterative solution of the necessary conditions. This objective is achieved, although strict optimality, based on absolute system performance and assurance of system stability, is sacrificed. Instead several suboptimal design procedures are developed, each based on minimizing the distance in some metric from the solution of a reference optimal problem. Two classes of control structure constraints are treated: (1) centralized output feedback, in which each control component is constrained to be a linear function of all of the output variables, and (2) decentralized output feedback, in which each control component is constrained to be a linear function of a prespecified, and possibly distinct, subset of the output variables. When the relative cost in the suboptimal problem is a quadratic functional, the algebraic necessary conditions that result, although still nonlinear and coupled, can be solved in closed form.

The essential features of the Kosut design methods are outlined in the remainder of Section 6.1. A careful analysis of the Kosut design approaches is given in Section 6.2. The key observation in this section is that certain assumptions relating to the sensor configuration upon which the published methods are based makes the methods inapplicable to most problems of large structure control. Motivated by this observation, extensions of the Kosut methods are developed which enable them to be used with arbitrary sensor

configurations. The Kosut methods, as extended, are successfully applied to the design of a vibration controller for a simple two body oscillator. Details of a nominal design, together with possible design alternatives, are outlined in Section 6.3. Conclusions, including recommendations for further study, are given in Section 6.4.

6.1.2 The Design Methods

6.1.2.1 Common Features

Each of the suboptimal design methods have certain features in common: one of several types of constraints are imposed *a priori* on the control law structure, and some optimal control problem for the plant of interest whose solution is known *a priori* is chosen as a reference.

Two classes of control structure constraints are considered. The first, centralized output feedback†, requires that elements of the controller output vector u(t) to the plant be constant linear combinations of elements of the plant output vector y(t)

$$u(t) = Gy(t) \quad , \quad G\colon m \times \ell \tag{6-1}$$

each of which are physically measurable constant linear combinations of elements of the plant state vector x(t)

$$y(t) = Cx(t) \quad , \quad C\colon \ell \times n, \ \ell \leq n \tag{6-2}$$

The feedback system structure is shown in Figure 6-1. The second, decentralized output feedback†, allows each component $u_i(t)$, i=1,...,m of the controller output vector to be a constant linear combination of possibly distinct subgroupings $y_i(t)$, i=1,...,m of the plant output vector

$$\left.\begin{aligned} u_i(t) &= g_i^T\, y_i(t), \quad g_i\colon \ell_i \times 1 \\ y_i(t) &= C_i\, x(t), \quad C_i\colon \ell_i \times n \end{aligned}\right\}\ \ell_i \leq n;\ i=1,\ldots,m \tag{6-3}$$

The feedback system structure, shown in Figure 6-2, generalizes the centralized case by allowing each control channel to have a different information structure.

In addition, an optimal control problem is formulated relative to the plant and initial conditions of interest

$$\dot{x}(t) = Ax(t) + Bu(t), \ t \geq 0, \ x(0) = x_0 \tag{6-4}$$

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†For descriptive purposes, the identifying labels used here differ from those used by Kosut.

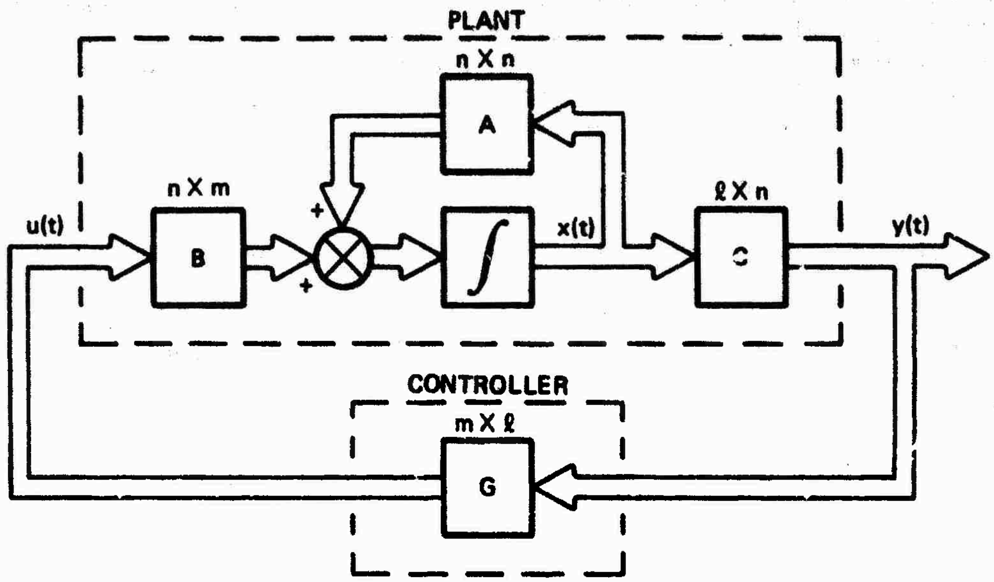

Figure 6-1. Centralized output feedback structure.

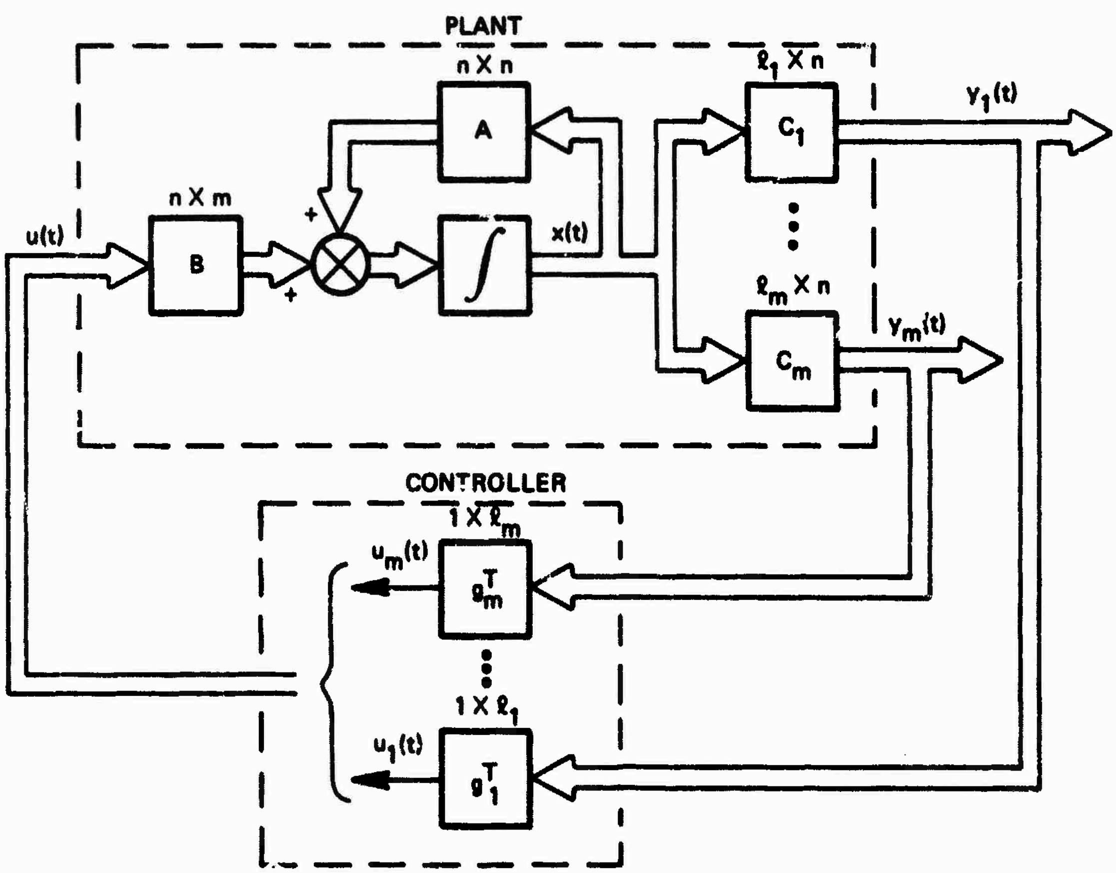

Figure 6-2. Decentralized output feedback structure.

and then solved for an optimal pair (x^*, u^*) to be used as a reference for subsequent suboptimal design. The nature of the cost criterion in this reference problem is free to be chosen by the designer. Whatever this choice may be, however, it is assumed that the optimal control is a constant linear combination of the optimal plant state

$$u^*(t) = F^*x^*(t) \quad , \quad F^*: m \times n \tag{6-5}$$

however, it is not assumed that F^* has a structure consistent with the control structure constraints above. Moreover, the optimal system is assumed to be asymptotically stable. As an example, the linear-quadratic problem for a time invariant plant satisfies these assumptions; in particular, the feedback matrix F^* in general is <u>not</u> consistent with the types of output feedback discussed.

The suboptimal design methods are distinguished by the way in which the controller gains are computed. Formulas for the gains are obtained from necessary conditions for a minimum in the suboptimal problem. The distinction between the two methods arises from different ways of penalizing the variation between controller gains in the suboptimal problem and those in the optimal reference problem. This variation can never be exactly zero, since the controller in the optimal reference problem is not required to satisfy control structure constraints.

6.1.2.2 Suboptimal Design by Minimum Error Excitation

The suboptimal system plant is described by Eq. (6-4), with each admissible control constrained to the form of Eq. (6-5)

$$u(t) = Fx(t) \quad , \quad F: m \times n \tag{6-6}$$

and such that the gain matrix F satisfies control structure constraints implied either by Eqs. (6-1) and (6-2), or by Eq. (6-3). It follows that each suboptimal trajectory $x(\cdot)$ corresponds to an error vector

$$e(t) \triangleq x(t) - x^*(t) \tag{6-7}$$

which satisfies the initial value problem

$$\dot{e}(t) = (A+BF)\, e(t) + B(F-F^*)x^*(t), \; t \geq 0; \; e(0) = 0 \tag{6-8}$$

The suboptimal cost measure is a quadratic functional weighting the forcing term in Eq. (6-8) with a positive definite matrix R

$$I_E(F) \triangleq \int_0^{+\infty} x^{*T}(t)(F-F^*)^T R(F-F^*)\, x^*(t)\, dt \tag{6-9}$$

The variational problem of minimizing $I_E(F)$ under the constraints described is converted to a mathematical programming problem by using in Eq. (6-9) the explicit solution form for $x^*(\cdot)$ obtainable from Eqs. (6-4) and (6-5). Since the optimal reference system is asymptotically stable, an integration by parts shows [7; p. 179] that minimizing Eq. (6-9) is equivalent to minimizing, over those gain matrices F that satisfy a control structure constraint, the expression

$$I_E(F) = x_0^T V x_0 \tag{6-10}$$

where V satisfies the matrix equation

$$(A+BF^*)^T V + V(A+BF^*) + (F-F^*)^T R(F-F^*) = 0 \tag{6-11}$$

In order to obtain results independent of the system initial state, the cost expression (6-10) is replaced by

$$\hat{I}_E(F,V) \triangleq \text{Trace } (V) \tag{6-12}$$

Under appropriate assumptions on the distribution of x_0 as a random variable, expressions (6-10) and (6-12) differ by a constant multiple [4].

Using standard mathematical programming techniques†, necessary conditions for a minimum are obtained, consisting of Eq. (6-11), a Lyapunov-type equation for the multiplier matrix P associated with the constraint Eq. (6-11)

$$(A+BF^*)P + P(A+BF^*)^T + I = 0 \tag{6-13}$$

and explicit expressions for the suboptimal gain matrix, F. For the case of centralized output feedback: $F = GC$, with

$$G = F^*[PC^T(CPC^T)^{-1}] \tag{6-14a}$$

For the case of decentralized output feedback: $F = \text{col}(F_1,\ldots,F_m)$, and each row vector $F_j = g_j^T C_j$ is characterized by

$$g_j^T = F_j^*[PC_j{}^T(C_jPC_j{}^T)^{-1}],\ j=1,\ldots,m \tag{6-14b}$$

where F_j^* is the j^{th} row vector of the matrix F^*.

†Development of these and subsequent necessary conditions are greatly facilitated by using several properties of the trace operator; these have been collected in Section 6.5.1.

Since F* is known in advance, Eqs. (6-13), (6-14), and (6-11) can be solved in succession for P, F, and V, respectively, instead of having to be solved by iteration. This was a major goal of the suboptimal design. Some further observations are also worth noting. First, since F* corresponds to an asymptotically stable closed loop optimal system, the matrix P obtained from Eq. (6-13) is positive definite [7; p. 254]. This is not sufficient, however, to guarantee invertibility of CPC^T, as required by Eqs. (6-14), without certain assumptions on the sensor matrix C. Kosut makes certain restrictive assumptions which guarantee invertibility of CPC^T, but does not discuss the general case. This topic is explored in depth in Section 6.2.3. Second, the expression (6-14b) depends on the assumption that the error excitation weighting matrix R in Eq. (6-9) is diagonal. This dependence is obscured since the elements of R do not appear in Eq. (6-14b); without such an assumption, Eq. (6-14b) must be replaced by the more general system

$$\sum_{j=1}^{m} \left[g_j^T C_j P C_k^T - F_j^* P C_k^T \right] r_{jk} = 0, \; k=1,\ldots,m$$

which involves the elements of R explicitly. In contrast, Eq. (6-14a) does not depend on such an assumption.

6.1.2.3 Suboptimal Design by Minimum Norm

The plant description and control constraints for this method are the same as those for the minimum error excitation approach. The suboptimal cost measure is the Euclidean distance between the equivalent state-feedback gain matrices in the suboptimal system [Eq. (6-6)] and the optimal reference system [Eq. (6-5)]

$$I_N(F) \triangleq \| F-F^* \| \equiv \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} (f_{ij} - f_{ij}^*)^2} \tag{6-15}$$

Minimization of I_N over matrices satisfying control structure constraints is a simple mathematical programming problem. It can be cast in terms of the trace operator, since, for any m×n matrix Γ

$$\|\Gamma\|^2 = \text{Trace } (\Gamma^T\Gamma)$$

Necessary conditions for a solution consist simply of explicit expressions for the suboptimal gain matrix, F. For the case of centralized output feedback: F = GC, with

$$G = F^*[C^T(CC^T)^{-1}] \tag{6-16a}$$

For the case of decentralized output feedback: $F = \mathrm{col}(F_1,\ldots,F_m)$, and each row vector $F_j = (g_j)^T C_j$ is characterized by

$$g_j = F_j^*[C_j^T(C_jC_j^T)^{-1}],\ j=1,\ldots,m \tag{6-16b}$$

6.2 Discussion

6.2.1 Advantages

The Kosut design methods are appealing for at least three reasons: (1) they are simple, (2) they are noniterative, thereby avoiding convergence difficulties, and (3) they can be used in conjunction with iterative optimal output feedback methods to improve convergence.

These methods share the simplicity inherent in all output feedback methods; i.e., by constraining the control law to be a function only of system variables that can be measured, the need for reconstruction or estimation of the full state is avoided. This keeps the order of the overall closed loop system relatively small, reduces the complexity of the analysis and design problems, and alleviates the computational difficulties (e.g., phasing and transport lag) associated with implementation in a real control system. In these methods, simplicity is further enhanced by constraining the output-control relation to be linear and time invariant, consistent with the corresponding nature of the plant considered.

A major goal achieved by the design methods is the development of explicit expressions for the controller gains (assuming the existence of inverses for certain matrices). The algebraic Eqs. (6-11), (6-13), and (6-14) that arise in the minimum error excitation method are similar in structure to those that occur in studies of optimal output feedback [4]; however in the latter problem, the equations do not in general admit a closed form solution. Moreover, algorithms proposed for iterative solution do not guarantee convergence to a solution [4]; in fact, experience with them has shown that in system ose order is much larger than the number of available outputs, satisfactory convergence is not obtained [5]. This situation is largely due to a lack of knowledge as to where to start the iteration. Convergence questions do not arise in direct application of the Kosut methods.

It has recently been observed that the Kosut method of minimum error excitation can assist the designer interested in an optimal output feedback approach [4]. By using the explicit solutions of Eqs. (6-11), (6-13) and (6-14) as an initial point for starting the iterative solution of the corresponding optimum output feedback equations, good convergence has been demonstrated [8].

6.2.2 Disadvantages

The Kosut design methods have at least three clear weaknesses: (1) no information on closed loop stability is available; (2) the results, as published, cannot be used when certain matrix products involving the sensor

matrix C [Eq. (6-2)] are not invertible; and (3) controller designs are not assured to have the desired "minimum-distance" property relative to the optimal reference problem chosen.

The most serious deficiency of the design methods is the lack of assurance that the controller gains obtained will lead to a stable closed loop system. This is immediately evident by observing that neither of the suboptimal cost measures [Eq. (6-9) or (6-15)] involve trajectories of the suboptimal system. In contrast, the Levine-Athans optimal output feedback method [4] does guarantee stability of the closed loop system, although, as noted above, iterative solution for the controller gains is usually required. We have observed that there is a direct connection between the achievement of a noniterative solution for the controller gains and the lack of stability information in Kosut's minimum error excitation method. In fact, his development can be modified in a natural way so as to obtain corresponding results which do contain stability information; however, the controller gain equations that result require iterative solution in general, and so little would be gained by using such an alternate approach. This alternate development of minimum error excitation is outlined in Section 6.5.2.

Each of the expressions for the controller gains, Eqs. (6-14) or (6-16), assumes the invertibility of certain matrix products, all of which have the general form $C\pi C^T$, where C is a sensor matrix with at least as many columns as rows, and π is positive definite†. Such a matrix product is invertible if and only if C has maximum rank [Section 6.2.3.2; Theorem 6-1]. Kosut makes the assumption, which will be seen to be unduly restrictive, that the system outputs are a subset of the physically measurable system states. Hence, the sensor matrix C is such that, by rearrangement of columns, it takes the form $[I_\ell \vdots 0]$, and thus has maximum rank. This assumption is not always satisfied for certain systems of interest: in particular, for reduced order models of large space structures [Section 6.2.4]. For such systems, the Kosut design methods, in their present form, simply cannot be used.

The familiar, but subtle, logic used in the development of the design methods should be carefully noted. The equations which identify the controller gains and related variables are first order necessary conditions based on the assumption of a minimum in the mathematical programming problem that is used as a representation of one of the suboptimal variational problems. It is worth observing that these necessary conditions have unique solutions whenever they are well defined†. However, such solutions represent only stationary interior points of the domain of the augmented cost function. There is no assurance that they correspond to a minimum in the suboptimal problem. If they do not (e.g., if they correspond to a maximum), poor alignment with the optimal reference problem, and unsatisfactory system performance using the associated controller design, may result.

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†Since A+BF* has been assumed to be asymptotically stable, positive definiteness of the matrix P in Eq. (6-14) and uniqueness of solutions for V and P in Eqs. (6-11) and (6-13), respectively, follow from the properties of the Lyapunov Eq. (6-13).

6.2.3 Extensions

6.2.3.1 Motivation. In each of the Kosut design methods, formulas for calculating the suboptimal controller gain matrix assume implicitly that matrix products of the form $C\pi C^T$, where π is positive definite and C is a sensor matrix, are invertible. Conditions for invertibility are not given; moreover, there is no discussion of possible alternative approaches for determining an appropriate controller gain matrix when the published formulas are inapplicable. Sensor configurations which do not have this invertibility property occur frequently [Section 6.2.4]. The control designer then faces two alternatives: either change the sensor configuration such that the required invertibility property is attained, or abandon the Kosut methods altogether. This is a serious drawback to the Kosut methods as published.

In this section we show that the Kosut methods can be extended so as to be usable with sensor configurations lacking invertibility. In each such case, a family of controller gain matrices can be found which satisfy the first order necessary conditions in the suboptimal problem. This gives the designer increased flexibility which can be utilized to improve the system performance.

6.2.3.2 Theoretical Results

In each Kosut design approach, the underlying necessary condition for suboptimality (not stated explicitly by Kosut) which generates an appropriate controller gain matrix is a linear matrix equation of the form

$$XA = B \tag{6-17}$$

The matrices A and B are products, known a priori, with the special structure

$$A = C\pi C^T \quad , \quad B = F^*\pi C^T \tag{6-18}$$

where $\pi:\nu\times\nu$ is positive definite, $C:\lambda\times\nu$ is a sensor matrix and $F^*:\mu\times\nu$ is unrestricted. We study solutions of Eq. (6-17) when the product matrix represented by A is not invertible.

First, conditions for invertibility of A are stated.

Theorem 6-1. Assume that π is positive definite†. The matrix product $C\pi C^T$ is invertible if and only if rank (C) = λ.

This result clarifies the conditions under which the Kosut methods, in their present form, are applicable; namely, the appropriate sensor matrix (either of a single channel in a decentralized system, or the full sensor

†An assumption of invertibility for π is not strong enough. Consider

$$\pi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = [0\ 1]$$

matrix of a centralized system) must be of maximum rank. It is recognized that a sensor configuration for which rank (C) $< \lambda$ can always be treated by reducing the number of output measurements (i.e., effectively reducing the number of sensors) so that the resulting sensor matrix C' is of maximum rank, with rank (C') = rank (C). However, such an approach will in general lead to a deterioration in system performance which may be intolerable. It is therefore desirable to be able to work with arbitrary sensor configurations. Theorem 6-1 is a special case of the following more general result.

Theorem 6-2. Assume that π is positive definite. Then

$$\text{rank } (C\pi C^T) = \text{rank } (C)$$

This result connects the deficiency in rank of the matrix A in the controller gain matrix Eq. (6-17) with the deficiency in rank of the sensor matrix. It will be seen below that the rank deficiency of the former determines the number of free parameters in a solution for the controller gain matrix. Theorem 6-2 is proved in Section 6.5.3.

Next, the general structure of solutions for matrix equations of the form Eq. (6-17) is briefly reviewed. The results to be stated can be deduced from corresponding results for the more general equation $AX - XB = C$ [9; Chapter 8]. However, results and proofs take a much simpler form for the special Eq. (6-17); they reduce to well-known results for the case when X and B are row vectors. Proofs are briefly outlined in Section 6.5.3.

Structure for solutions of the homogeneous equation $XA = 0$ is given by Theorem 6-3. At least one solution, the zero matrix, always exists.

Theorem 6-3. Let $A{:}\lambda\times\nu$, $X^0{:}\mu\times\lambda$ be matrices. Denote $r \triangleq$ rank (A). Then:

(1) X^0 satisfies the homogeneous equation $XA = 0$ if and only if X^0 is a product of the form ΓS, where $\Gamma \equiv [0 \vdots \Gamma']$ is a $\mu\times\lambda$ matrix whose first r columns are zero, and whose last λ-r columns are arbitrary, and S is a nonsingular $\lambda\times\lambda$ matrix such that SA is in row-echelon form; and

(2) The zero solution $X^0 = 0$ is unique if and only if rank (A) $= \lambda$.

This result shows that when $r \equiv$ rank (A) $< \lambda$, there are $\mu(\lambda-r)$ arbitrary parameters in the general solution. Solutions of the nonhomogeneous equation $XA = B$, if any exist, have the structure given by Theorem 6-4.

Theorem 6-4. Let $A{:}\lambda\times\nu$, $B{:}\mu\times\nu$, and $X^0{:}\mu\times\lambda$ be matrices. Then:

(1) X^0 satisfies the equation $XA = B$ if and only if it is a sum of the form $X_p + X_c$, where X_p satisfies $XA = B$ and X_c satisfies $XA = 0$; and

(2) Equation $XA = B$ has at most one solution if and only if rank (A) $= \lambda$.

This result makes no assertion about existence of solutions. Existence criteria are given by Theorem 6-5.

Theorem 6-5. Let $A:\lambda\times\nu$, $B:\mu\times\nu$ be matrices. Then the equation $XA = B$ has a solution if and only if rank $\begin{bmatrix} A \\ \hline B \end{bmatrix}$ = rank A.

The main result of this section is Theorem 6-6, which establishes that conditions for existence of solutions are always satisfied by the controller gain matrix Eqs. (6-17), (6-18) that underlie the Kosut methods.

Theorem 6-6. Let $\pi:\nu\times\nu$, $C:\lambda\times\nu$, $F^*:\mu\times\nu$ be matrices with π positive definite. Then:

$$\text{rank} \begin{bmatrix} C\pi C^T \\ \hline F^*\pi C^T \end{bmatrix} = \text{rank } (C\pi C^T)$$

Proof of this result is given in Section 6.5.3.

6.2.3.3 Implications

The theoretical results have the following implications for the control system designer.

(1) The Kosut suboptimal design methods, as extended, are applicable for all sensor configurations.

(2) If a sensor matrix C† has maximum rank, then the matrix $C\pi C^T$ is invertible [Theorem 6-1], and the equation

$$G\ (C\pi C^T) = F^*\pi C^T \tag{6-19}$$

has the unique solution $G^0 = F^*\pi C^T(C\pi C^T)^{-1}$ [Theorem 6-4] for the controller gain matrix.

(3) If a sensor matrix C† has rank ρ less than full rank (λ), then the matrix $C\pi C^T$ has the same rank deficiency, $\lambda - \rho$ [Theorem 6-2]. Nevertheless, Eq. (6-19) is algebraically consistent [Theorems 6-5, 6-6], so one particular solution G^0 exists. The general solution to Eq. (6-19) has the form

$$G = G^0 + \Gamma S \tag{6-20}$$

† $C:\lambda\times\nu$, $\lambda \leq \nu$.

where $\Gamma \equiv [0 \vdots \Gamma']$ has the first ρ columns zero, with Γ' a matrix of $\mu(\lambda-\rho)$ arbitrary parameters, and S is a nonsingular matrix (nonunique) for which $S(C\pi C^T)$ is in row-echelon form [Theorems 6-3, 6-4].

(4) As stated earlier, Eq. (6-19) is a necessary condition for obtaining a "closeness" in some suboptimal sense to an optimal reference problem with state feedback gain F*. The free parameters appearing in the solution (6-20) for the controller gain matrix give the control designer extra flexibility which may be used to improve performance in some specific way (e.g., increase damping of one or more modes), or decrease side effects of the design (e.g., alleviate control spillover related to a reduced order controller design). This shows in a general way a connection between an increase in the number of sensors (decreased rank of a sensor matrix) and improved performance based on design by the Kosut methods. These properties are illustrated in detail in the example of Section 6.3.

(5) Equations of the form (6-19) also arise in the study of optimal constant gain output feedback [4], both as part of a set of coupled necessary conditions for optimality, and as part of an algorithm for numerical solution of the necessary conditions. The results reported here shed additional light on the existence and properties of solutions for these coupled systems.

6.2.4 Applicability to Large Space Structure Control

Design of controllers for real large space structures almost inevitably requires use of a relatively low order structural model (design model) in the preliminary design process, with evaluation of the design model against a higher order, but still finite-dimensional model (evaluation model), as discussed in Section 2.2.3. We make several important observations regarding the use of the Kosut design methods with a reduced order structural model: (1) sensor matrices in the design model do not, in general, have maximum rank; (2) formulas for calculating the composite gain connecting the plant state and controller output with the design model cannot be used in the evaluation model.

As noted earlier [Section 6.2.2], the Kosut methods effectively require the system sensor matrix (or the sensor matrices in each control channel, in the decentralized case) to have maximum rank. We show here why such a condition cannot be expected to hold in a reduced order structural model. Consider a finite dimensional structural model (e.g., via finite elements) in terms of physical coordinates q with sensor matrix $C:\ell\times 2n$, $\ell \leq 2n$

$$M\ddot{q} + Kq = F_A u \tag{6-21}$$

$$y = C \begin{bmatrix} q \\ \cdots \\ \dot{q} \end{bmatrix} \tag{6-22}$$

Accepting Kosut's assumption here that each output element y_i is one of the components of the state vector $(q,\dot{q})$, the matrix C has maximum rank ℓ; in fact, C is a column permutation of the matrix $\left[I_\ell \vdots 0\right]$. Transformation of Eqs. (6-21), (6-22) to modal coordinates via the nonsingular matrix Φ (i.e., $q = \Phi\eta$) leads to the system

$$\ddot{\eta} + \Omega^2 \eta = \Phi^T F_A u,$$

$$y = C \begin{bmatrix} \Phi & \vdots & 0 \\ \hline 0 & \vdots & \Phi \end{bmatrix} \begin{bmatrix} \eta \\ \hline \dot{\eta} \end{bmatrix} \equiv C\hat{\Phi} \begin{vmatrix} \eta \\ \hline \dot{\eta} \end{vmatrix} \tag{6-23}$$

which still preserves the rank of the new sensor matrix $C\hat{\Phi}$. After selection of the critical modes to be included in a design model, the modal state vector $(\eta,\dot{\eta})$ may be reordered by a (nonsingular) row interchange matrix S

$$S \begin{bmatrix} \eta \\ \hline \dot{\eta} \end{bmatrix} = \begin{bmatrix} x_c \\ \hline x_R \end{bmatrix} \tag{6-24}$$

where $x_c : \nu \times 1$ is the desired state vector for the design model. The sensor matrix $\hat{C} \equiv C\hat{\Phi}S^{-1} \equiv [C_c \vdots C_R]$ relative to the state (x_c, x_R)

$$y = \hat{C} \begin{bmatrix} x_c \\ \hline x_R \end{bmatrix} = C_c x_c + C_R x_R \tag{6-25}$$

retains maximum rank. However, the sensor matrix C_c relative to the reduced state x_c in the design model is simply the submatrix consisting of the first ν columns of $\hat{C}$, and therefore, in general

$$\text{rank } (C_c) \leq \text{rank } (\hat{C}) \tag{6-26}$$

Since there is, in general, no relationship between the indices that tag the critical elements of the modal state $(\eta,\dot{\eta})$ and those that tag the ℓ linearly independent columns of the matrix $C\hat{\Phi}$, inequality [Eq. (6-26)] may well be strict, as in the example below [Section 6.3].

The next observation, though self evident, is worth pointing out. The Kosut design methods give explicit expressions for the plant state to controller output gain matrix F [Eq. (6-6)], rather than for the plant output to controller output matrix G [Eq. (6-1) for the centralized case]. Having designed the controller with a reduced order design model, Eq. (6-6) may be written

$$u(t) = F_c x_c(t) \tag{6-27}$$

Any expression of the form

$$u(t) = [F_c \,\vdots\, F_R] \begin{bmatrix} x_c(t) \\ \hdashline x_R(t) \end{bmatrix} \tag{6-28}$$

for use in the evaluation model, where F_R is unspecified, is consistent with Eq. (6-27). One appropriate choice for F_R can be made by noting that Eq. (6-27) has the equivalent form (centralized case)

$$u(t) = [GC_c]\, x_c(t) = G[C_c x_c(t)] \tag{6-29}$$

and then adjusting Eq. (6-29) so as to apply the matrix G to the <u>actual</u> system output

$$u(t) = G[C_c x_c(t) + C_R x_R(t)] = G\, y(t) \tag{6-30}$$

Other choices for F_R could certainly be made; however any systematic selection procedure would be essentially equivalent to including the variables x_R in the design model, which runs counter to their definition as variables <u>excluded</u> from the design model.

One additional brief comment: as noted above, the Kosut methods do not guarantee stability for the controller in the design model. However, even if a reduced order controller turns out to be stable, nothing can be concluded <u>a priori</u> about the stability of the evaluation model driven by such a controller.

6.3 Example

Kosut's method of minimum error excitation is used to design a second order controller for the fourth order spring mass system example, which is described along with numerical data for parameters in **Section 2.** A centralized output feedback structure is imposed upon the controller. The principal design objective is to achieve a damping ratio of 0.1 in the critical mode corresponding to out-of-phase vibration of the two-mass system. The optimal reference system is of linear-quadratic type.

6.3.1 Controller Design for the Optimal Reference System

Choice of an optimal reference system for use with Kosut's suboptimal methods is at the discretion of the control system designer. A linear quadratic optimal control system with full state feedback is chosen because of its simplicity and the availability of closed form solutions. State vector differential equations for the critical mode in the form of a general second order system with two control inputs† may be written in the form [Eq. (6-4)]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_2^2 & -2\zeta_2\omega_2 \end{bmatrix}}_{A_c} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ \psi_1 & \psi_2 \end{bmatrix}}_{B_c} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{6-31}$$

where ω_2 and ζ_2 ($\equiv 0$) are the natural frequency and damping ratio, respectively, of the mode to be controlled. The cost functional to be minimized is

$$J(u) \triangleq \frac{1}{2} \int_0^{+\infty} \left[x^T(t)Q\,x(t) + u^T(t)Ru(t)\right] dt \tag{6-32}$$

with R positive definite, Q positive semi-definite, and $x \equiv \mathrm{col}(x_1,x_2)$, $u \equiv \mathrm{col}(u_1,u_2)$. The solution to this problem for a general n^{th} order system, assuming (A_c, B_c) is completely controllable, is well known [1; Ch. 9]. The optimal control is a linear function of the state [Eq. (6-5)]

$$u^*(t) = \left[-R^{-1}B_c^T K\right] x^*(t) \equiv F^*\, x^*(t) \tag{6-33}$$

where K is the (unique) positive definite solution of the algebraic Riccati equation:

$$KB_c R^{-1} B_c^T K - A_c^T K - KA_c - Q = 0 \tag{6-34}$$

---

†Appearance of two control inputs for a single second order system arises due to the truncation of the full modal system to form the second order design model.

Moreover, the closed loop optimal system is stable. Relationships between the natural frequency, ω^*, and the damping ratio, ζ^*, of the second order closed loop optimal system and the parameters $(A_c, B_c; Q, R)$ of the open loop optimization problem are also well known [12], [13].† They are obtained by finding the coefficients of the closed loop characteristic equation

$$\det (A_c + B_c F^* - \lambda I) = 0 \tag{6-35}$$

and may be expressed in the form (modified for the case of two inputs)

$$q_{11}\, \psi^T R^{-1} \psi = (\omega^*)^4 - \omega_2^4 \tag{6-36}$$

$$\frac{1}{4} q_{22}\, \psi^T R^{-1} \psi = (\omega^*)^2 \left[(\zeta^*)^2 - \frac{1}{2}\right] + \omega_2^2 \left[\zeta_2^2 - \frac{1}{2}\right] \tag{6-37}$$

where $Q \equiv [q_{ij}]$, and ψ^T is the non-zero row vector of B_c in Eq. (6-31). For the purpose of achieving prescribed values for ω^* and ζ^*, there is no loss of generality in assuming Q to be diagonal and R to be the identity. Denoting by ζ_{2D} the desired damping ratio ($\equiv 0.1$) for the controlled mode in the final reduced order design, the following characteristics are prescribed for the optimal reference system.

$$\omega^* \triangleq (1 + \alpha)\omega_2, \quad \alpha = 0.005 \qquad \text{(6-38)}^{\dagger\dagger}$$

$$\zeta^* \triangleq \zeta_{2D} \left(\frac{\omega_2}{\omega^*}\right)\left(\frac{1 + (\omega^*)^2}{1 + \omega_2^2}\right) \qquad \text{(6-39)}^{\dagger\dagger}$$

Based on these prescriptions, the corresponding diagonal elements of Q are computed from Eqs. (6-36) and (6-37)

$$Q \equiv \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} = \begin{bmatrix} 1.0436469 & 0 \\ 0 & 0.15959926 \end{bmatrix}$$

---

†Documented relationships assume a single input — single output second order system. Modification to apply to the dual input situation here is trivial.

††The form of Eq. (6-38) is an arbitrary choice, with $\omega^* \neq \omega_2$, that satisfies the requirement that $q_{11} \geq 0$. The form of Eq. (6-39) ensures that $\zeta_2 = \zeta_{2D}$ will hold exactly in the final design.

Equation (6-34) for the symmetric matrix $K \equiv [k_{ij}]$ reduces to the system

$$k_{12}^2 (\psi^T\psi) + 2k_{12}\,\omega_2^2 + q_{11} = 0$$

$$k_{12}\,k_{22}\,(\psi^T\psi) + k_{22}\omega_2^2 - k_{11} = 0$$

$$k_{22}^2 (\psi^T\psi) - 2k_{12} - q_{22} = 0$$

The positive definite solution of this system is:

$$K \equiv \begin{bmatrix} k_{11} & k_{12} \\ & \\ k_{12} & k_{22} \end{bmatrix} = \frac{1}{(\psi^T\psi)} \begin{bmatrix} 2\zeta^*(\omega^*)^3 & (\omega^*)^2 - \omega_2^2 \\ & \\ (\omega^*)^2 - \omega_2^2 & 2\zeta^*\omega^* \end{bmatrix} = \begin{bmatrix} 4.0767481 & 0.077477583 \\ & \\ 0.077477583 & 0.60229008 \end{bmatrix}$$

The state-feedback gain matrix F* defined by Eq. (6-33) is then

$$F^* = -\begin{bmatrix} k_{12}\psi_1 & k_{22}\psi_1 \\ & \\ k_{12}\psi_2 & k_{22}\psi_2 \end{bmatrix} = \begin{bmatrix} 0\ 066389774 & 0.51609641 \\ & \\ -0.028241581 & -0.21954252 \end{bmatrix} \qquad (6\text{-}40)$$

One may verify that with this solution for F*, Eq. (6-35) has the form

$$\lambda^2 + 2\zeta^*\omega^*\lambda + (\omega^*)^2 = 0$$

6.3.2 Suboptimal Controller Design by Minimum Error Excitation

The structure of the suboptimal controller chosen for the second order design model Eq. (6-31) is centralized output feedback [Fig. 6-1]. First, we observe that with the specified two sensor configuration, the sensor matrix C_c for the design model has a rank deficiency of one; we trace the related discussion of Section 6.2.4. Referred to the physical variables $q = \text{col}\,(q_1,q_2)$, the full system sensor matrix C [Eq. (6-22)] for pure velocity sensing is $[0|I_2]$. Since mode 2 is the mode to be controlled, the row interchange matrix reordering the modal state vector $(\eta,\dot{\eta})$ into the critical-residual state vector partition (x_c,x_R) [Eq. (6-24)] is:

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The full system sensor matrix $\hat{C}$ referred to (x_c, x_R) [Eq. (6-25)] is

$$\hat{C} = \left[\begin{array}{cc:cc} 0 & \psi_1 & 0 & \phi_1 \\ 0 & \psi_2 & 0 & \phi_2 \end{array}\right] = \left[C_c \;\vdots\; C_R\right]$$

Hence [Eq. (6-26)]

$$\text{rank}\ (C_c) = 1 < 2 = \text{rank}\ (C)$$

The extension of Kosut's method developed in Section 6.2.3 allows the suboptimal design to continue in spite of this rank deficiency.

Next, we solve the necessary conditions for the suboptimal method of minimum error excitation. Using the known expression (6-40) for F^*, system (6-13) for the (symmetric) multiplier matrix $P = [p_{ij}]$ reduces to

$$2\ p_{12} = -1$$

$$-(\omega^*)^2\ p_{11} - 2\zeta^*\omega^*\ p_{12} + p_{22} = 0$$

$$-2(\omega^*)^2\ p_{12} - 4\zeta^*\omega^*\ p_{22} = -1$$

The solution is:

$$P = \begin{bmatrix} [\zeta^*\omega^* + h(\zeta^*,\omega^*)]/(\omega^*)^2 & -\frac{1}{2} \\ -\frac{1}{2} & h(\zeta^*,\omega^*) \end{bmatrix} = \begin{bmatrix} 1.1373892 & -0.5 \\ -0.5 & 7.4375655 \end{bmatrix} \qquad (6\text{-}41)$$

where $h(\zeta^*,\omega^*) \triangleq [1+(\omega^*)^2]/4\zeta^*\omega^*$. The gain equation (6-14a) must be replaced by its antecedent equation [Eq. (6-19)]

$$G\,(C_c\,PC_c{}^T) = F^*PC_c{}^T \tag{6-42}$$

Consistent with Theorem 6-6, we find that $F^*PC_c{}^T = -\sigma[C_cPC_c{}^T]$, where

$$\sigma = \frac{k_{12}\,p_{12} + k_{22}\,p_{22}}{p_{22}} = \frac{2\zeta^*\omega^*}{\psi^T\psi} \cdot \frac{(1+\omega_2{}^2)}{[1+(\omega^*)^2]} = 0.59708155 \triangleq \sigma^0 \tag{6-43}$$

System Eq. (6-42) reduces to the homogeneous equation

$$(G + \sigma\,I_2)\,C_c\,PC_c{}^T = 0 \tag{6-44}$$

Consistent with Theorem 6-3, system Eq. (6-44) has the following general solution

$$G(\varepsilon,\delta) = \begin{bmatrix} -\sigma + \varepsilon\,\psi_2/\psi_1 & -\varepsilon \\ \\ -\delta & -\sigma + \delta\,\psi_1/\psi_2 \end{bmatrix} \tag{6-45}$$

where ε and δ are arbitrary parameters.

A brief look at the closed loop dynamics for the design model reveals the following: (1) the overall gain matrix $F \triangleq GC_c$ is independent of the free parameters ε and δ

$$F = \begin{bmatrix} 0 & -\sigma\psi_1 \\ \\ 0 & -\sigma\psi_2 \end{bmatrix} \tag{6-46}$$

(2) the characteristic polynomial $\det(A_c + B_cF - \lambda I)$ is

$$\lambda^2 + \sigma(\psi^T\psi)\lambda + \omega_2{}^2$$

which shows that the system is stable, with the same natural frequency ω_2 as the open loop design model, and with damping ratio $\zeta = 0.1$ as desired [Eq. (6-39)]

$$\zeta = \frac{\sigma(\psi^T\psi)}{2\omega_2} = \zeta^*\,\frac{\omega^*}{\omega_2}\,\frac{(1+\omega_2{}^2)}{[1+(\omega^*)^2]} = \zeta_{2D} \equiv 0.1$$

This shows that the desired performance in the reduced order suboptimal design is a function of the parameter σ alone, and therefore [Eq. (6-43)] is achievable by judicious specification of parameters (ζ^*, ω^*) in the optimal reference problem. The free parameters have no influence upon the dynamics of the reduced order model in isolation; they may therefore be chosen so as to improve the system performance when the reduced order controller is connected to the full system.

6.3.3 Performance Evaluation of the Controller Design

6.3.3.1 Stability Analysis

The reduced order controller is correctly connected to the full system [Sec. 6.2.4] through the relation [Eq. (6-25)]

$$u(t) = G(\varepsilon,\delta)\ y(t) = G(\varepsilon,\delta)\hat{C}\begin{bmatrix} x_c(t) \\ \hline x_R(t) \end{bmatrix}$$

In general, stability of the full fourth order system matrix $A+B[G(\varepsilon,\delta)\hat{C}]^\dagger$ can be investigated using the Routh-Hurwitz criteria; however, the cubic polynomials in the pair (ε,δ) which appear limit the insight that can be obtained analytically. The stability analysis can be simplified considerably by constraining one of the available degrees of freedom. The differential equation for the critical mode η_2 in the full closed loop system is

$$\ddot{\eta}_2 + \sigma(\psi^T\psi)\dot{\eta}_2 + \omega_2^2\eta_2 = -[\sigma(\phi^T\psi) + (\delta-\varepsilon)\det\Phi]\dot{\eta}_1 \qquad (6\text{-}47)$$

The difference $\delta-\varepsilon$ may be chosen so that the coefficient of the modal cross-coupling term in Eq. (6-47) vanishes

$$\delta-\varepsilon = \frac{\sigma(-\phi^T\psi)}{\det\Phi} = 0.18649694 \qquad (6\text{-}48)$$

Enforcing this relationship makes the critical mode dynamics independent of excitations of the residual mode. In particular, the critical mode dynamics become identical to those in the reduced-order design. It also enables the full system equations to be expressed in closed form as a product of quadratic factors:

---

† The matrices A and B appearing here are the matrices of the full system written in the form of Eq. (6-4), with rows arranged consistent with the decomposition $x = \text{col}\ (x_c \vdots x_R)$ of the state vector.

$$\det(A + B[G(\delta-\varepsilon)\hat{C}] - \lambda I) =$$

$$\{\lambda^2 + \sigma(\psi^T\psi)\lambda + \omega_2^2\}\{\lambda^2 + [\sigma(\phi_1^2 + 2\phi_2^2) + \varepsilon\frac{\det^2\Phi}{(-\psi_1)\psi_2}]\lambda + \omega_1^2\} \quad (6\text{-}49)$$

By employing the weighted orthogonality properties of the modal transformation Φ,† the region of closed loop stability in the ε-σ parameter plane may be expressed simply by the conditions [Fig. 6-3]

$$\left.\begin{aligned} \sigma &> 0 \\ \varepsilon &> -2(-\psi_1)\psi_2\sigma \equiv -0.62469504\ \sigma \end{aligned}\right\} \quad (6\text{-}50)$$

The remaining degree of freedom (choice of ε) may be used to adjust the damping in the residual mode η_1. The differential equation for this mode in the full closed loop system is:

$$\ddot{\eta}_1 + [\sigma(\phi^T\phi) + (\delta\frac{\phi_2}{\psi_2} - \varepsilon\frac{\phi_1}{\psi_1})\det\Phi]\,\dot{\eta}_1 + \omega_1^2\eta_1 = -[\sigma(\phi^T\psi)]\dot{\eta}_2 \quad (6\text{-}51)$$

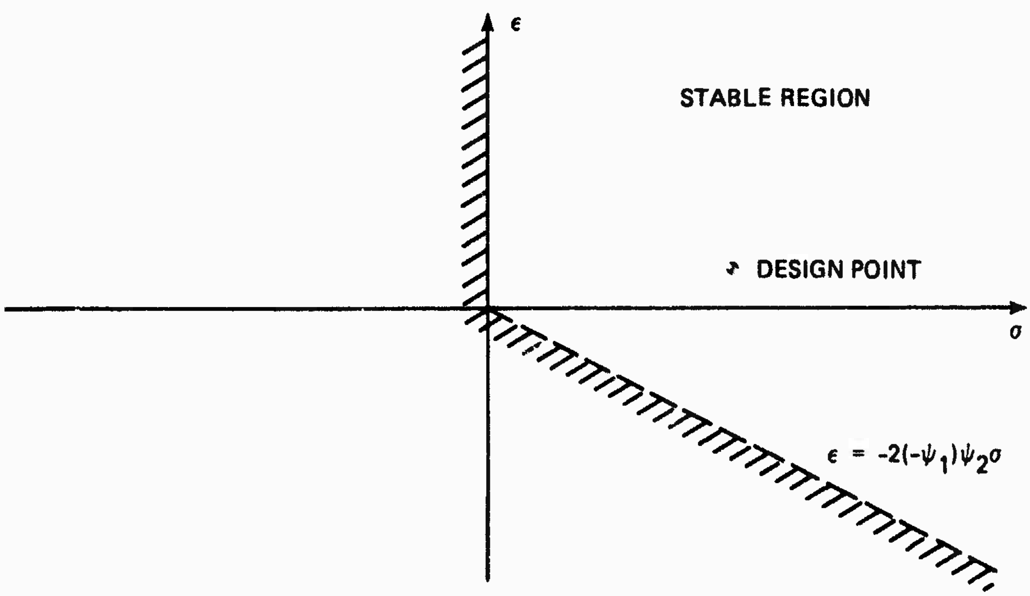

Figure 6-3. Region of stability for closed loop two-mass system.

† $\phi^T M\Phi = I$.

Incorporating the constraint relation [Eq. (6-48)], this reduces to

$$\ddot{\eta}_1 + [\sigma(\phi_1^2 + 2\phi_2^2) + \varepsilon \frac{\det^2\Phi}{(-\psi_1)\psi_2}]\, \dot{\eta}_1 + \omega_1^2\eta_1 = -[\sigma(\phi^T\psi)]\dot{\eta}_2 \qquad (6\text{-}52)$$

One may verify that the coefficient of $\dot{\eta}_1$ in Eq. (6-52) is positive for all pairs (σ,ε) satisfying the stability conditions [Eq. (6-50)]. We choose ε so as to obtain "optimal" damping $\zeta_1 = \zeta_{1D} \triangleq 1/\sqrt{2}$ of the residual mode

$$\varepsilon^0 = \left[\zeta_{1D} - \frac{\sigma}{2\omega_1}\right] \frac{2\omega_1(-\psi_1)\psi_2}{\det^2\Phi} = 0.10963135 \qquad (6\text{-}53)$$

The design point corresponding to the parameters $\sigma^0,\varepsilon^0,\delta^0$ fixed by Eqs. (6-43), (6-48), (6-53) is depicted on Figure 6-3; the corresponding gain matrix $G(\varepsilon^0,\delta^0)$ [Eq. (6-45)] is

$$G(\varepsilon^0,\delta^0) = \begin{bmatrix} -0.64371769 & -0.10963135 \\ -0.29612829 & -1.2932143 \end{bmatrix}$$

6.3.3.2 Spillover Effects

Controller design using the Kosut methods does not eliminate control or observation spillover associated with a reduced order design. However, the constraint (6-48) imposed on the two free design parameters in this example prevents the residual mode dynamics from feeding back into the critical mode dynamics. This is clearly seen from the system equations (6-47) and (6-51), which reduce, at the design point $(\sigma^0,\varepsilon^0,\delta^0)$, to

$$\ddot{\eta}_2 + 2(0.1)\omega_2\dot{\eta}_2 + \omega_2^2\,\eta_2 = 0 \qquad (6\text{-}54)$$

$$\ddot{\eta}_1 + 2(\tfrac{1}{\sqrt{2}})\omega_1\dot{\eta}_1 + \omega_1^2\eta_1 = -[\sigma^0(\phi^T\psi)]\dot{\eta}_2 \qquad (6\text{-}55)$$

External disturbances which initially perturb only the residual mode will be damped "optimally" ($\zeta_1 = 1/\sqrt{2}$) without exciting the critical mode at all [Fig. 6-4]. External disturbances which perturb the critical mode will be damped slowly ($\zeta_2 = 0.1$); these disturbances cause residual mode excitation, but such excitation does not affect subsequent critical mode dynamics [Fig. 6-5]. Such partial decoupling of the modal dynamics significantly alleviates the usual concern over control spillover. As a specific example, residual mode response in the system Eq. (6-54), Eq. (6-55) to an initial disturbance is a damped oscillation of form:

$$\eta_1(t) = e^{-d_1 t}\ \mathrm{Re}\ \{\alpha_1 e^{i\Omega_1 t}\} + e^{-d_2 t}\ \mathrm{Re}\ \{\alpha_2 e^{i\Omega_2 t}\} \tag{6-56}$$

where the α_j are complex functions of the initial conditions and $\lambda_j = -d_j \pm i\Omega_j$ are the system eigenvalues, j=1,2. The modal response to an initial perturbation in the residual mode ($\eta_1(0)=1$, $\dot{\eta}_1(0)=0$, $\eta_2(0)=\dot{\eta}_2(0)=0$) is shown in Figure 6-4. The apparent excitation of the critical mode shown is due to numerical roundoff error associated with the transformation from physical to modal coordinates. The modal response to an initial perturbation in the critical mode ($\eta_1(0)= \dot{\eta}_1(0)=0$, $\eta_2(0)=1$, $\dot{\eta}_2(0)=0$) is shown in Figure 6-5. It is recognized that initial conditions would normally be specified in terms of physical, rather than modal, coordinates. The use of "modal" initial conditions here is for the purpose of demonstrating specific spillover effects.

Observation spillover terms contaminate the measurements whenever $\dot{\eta}_1 \neq 0$. As can be seen from the foregoing remarks, this condition is, in a practical sense, quite short lived, and the effects on the system are relatively benign.

6.3.3.3 <u>Specific Response Characteristics</u>. When referred to the physical coordinates q_1 and q_2, the system response to nonzero initial conditions has the same general form as the modal response (6-56). Simulated response to the initial conditions $q_1(0)=\dot{q}_1(0)=0$, $q_2(0)=1$, $\dot{q}_2(0)=0$ is shown in Figure 6-6. The response indicates quite satisfactory speed of response, acceptable overshoot, and an asymptotic approach to zero for each physical coordinate. Modal characteristics of the same response are shown in Figure 6-7.

Response to a periodic disturbance $e^{i\omega t}$ in either physical coordinate q_j is characterized by a "steady state" response of form $A_j(\omega)e^{i\omega t}$; i.e., the difference $q_j(t) - \mathrm{Re}\ \{A_j(\omega)e^{i\omega t}\}$ decays to zero in the manner of Eq. (6-56). Evaluated at the design point, the complex amplitude function $A_2(\omega)$ is

$$A_2(\omega) = \frac{Q_2(j\omega)}{F_2(j\omega)} = \frac{0.5\ (j\omega)^2 + 0.32185884\ (j\omega) + 2.5}{(j\omega)^4 + 1.2903248\ (j\omega)^3 + 7.4\ (j\omega)^2 + 5.3319905\ (j\omega) + 2}$$

where $Q_2(s)$ and $F_2(s)$ are the Laplace transforms of the coordinate q_2 and the disturbance force f_2 on mass 2, respectively. Plots of $A_2(\omega)$ are shown in Figure 6-8.

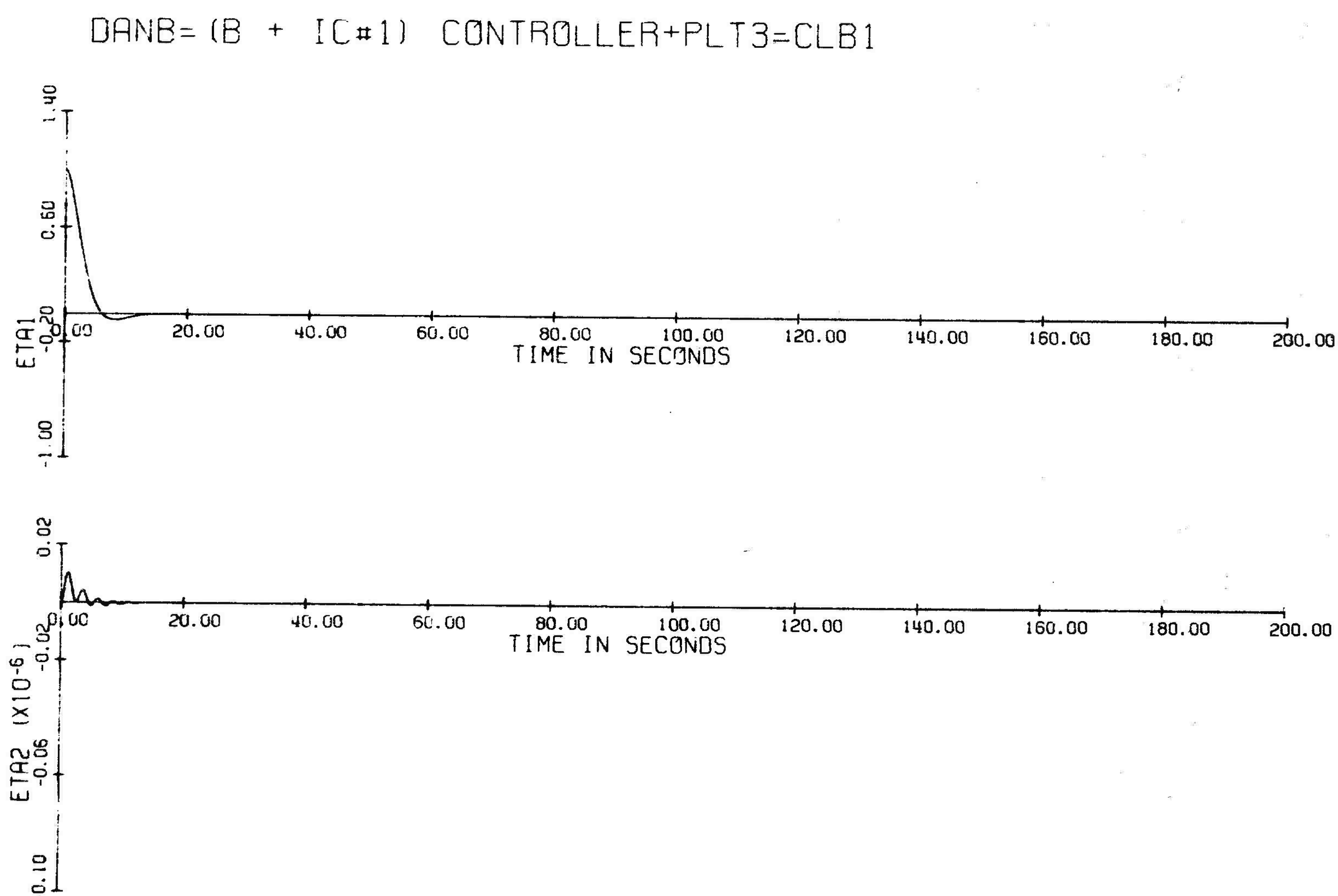

Figure 6-4. Modal response to initial conditions $\eta_1(0) = 1$, $\dot{\eta}_1(0) = 0$, $\eta_2(0) = \dot{\eta}_2(0) = 0$ (two sensor configuration).

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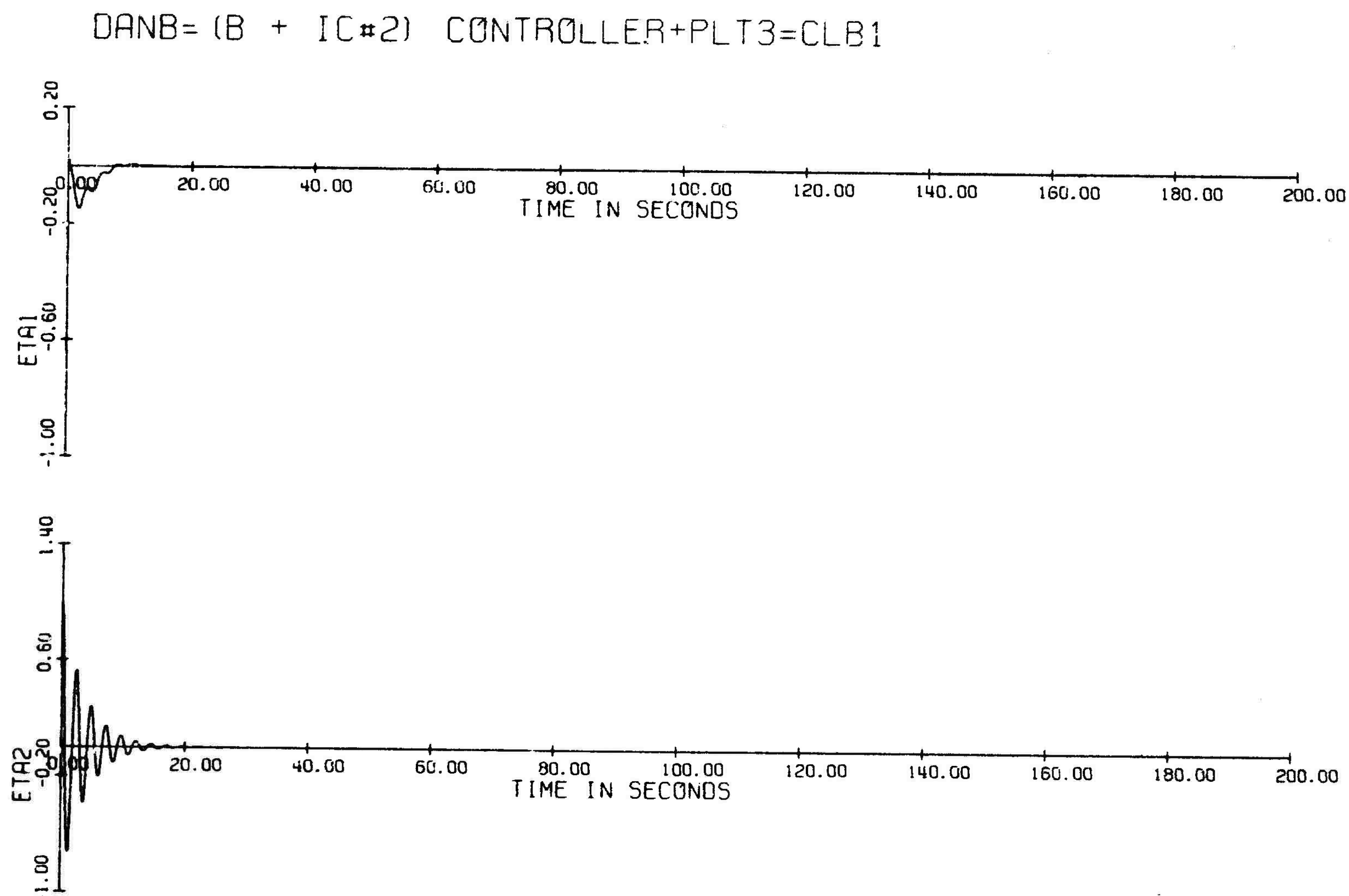

Figure 6-5. Modal response to initial conditions $\eta_1(0) = \dot{\eta}_1(0) = 0$, $\eta_2(0) = 1$, $\dot{\eta}_2(0) = 0$ (two sensor configuration).

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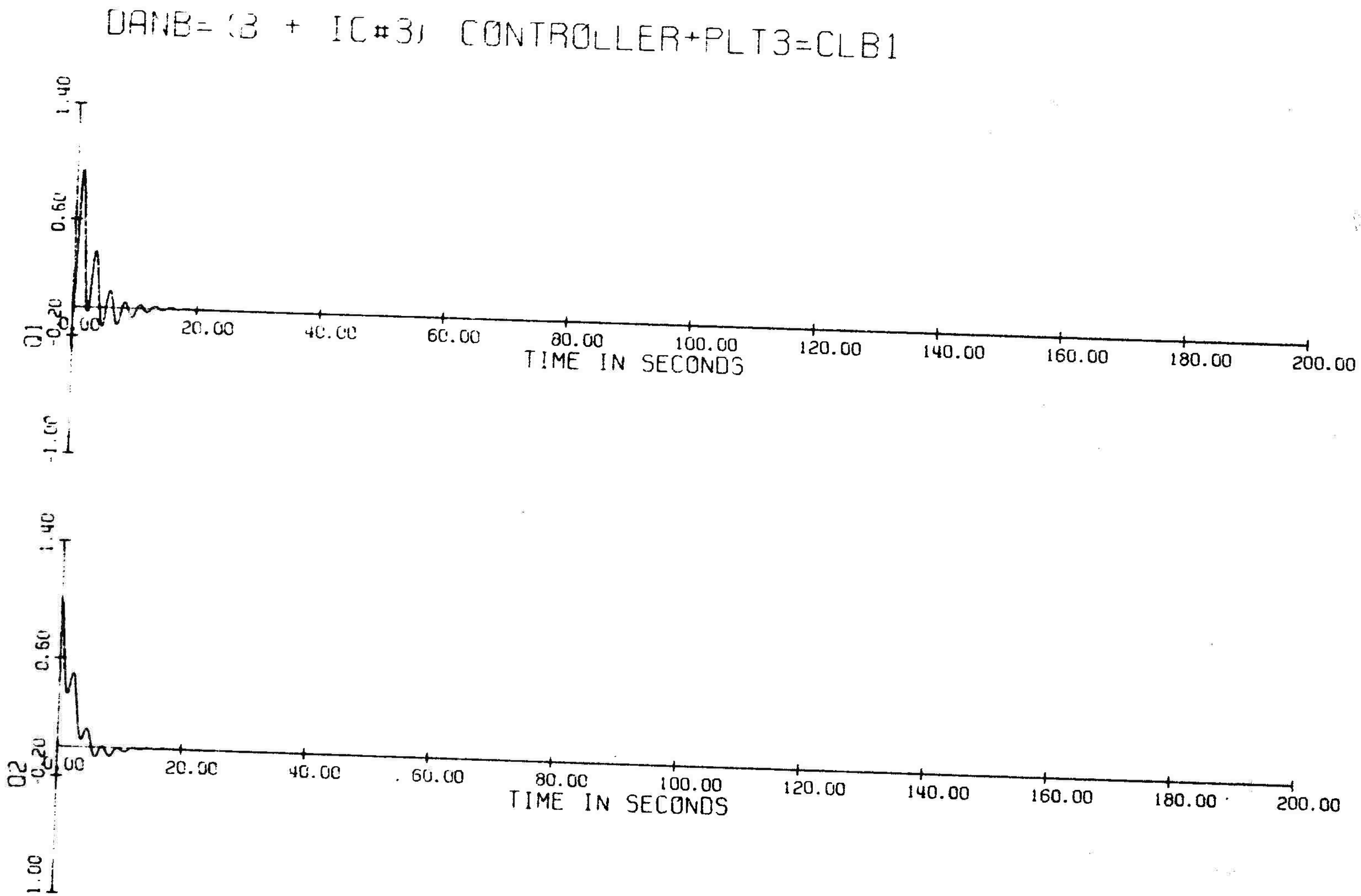

Figure 6-6. Response to initial conditions $q_1(0) = \dot{q}_1(0) = 0$, $q_2(0) = 1$, $\dot{q}_2(0) = 0$ (two sensor configuration).

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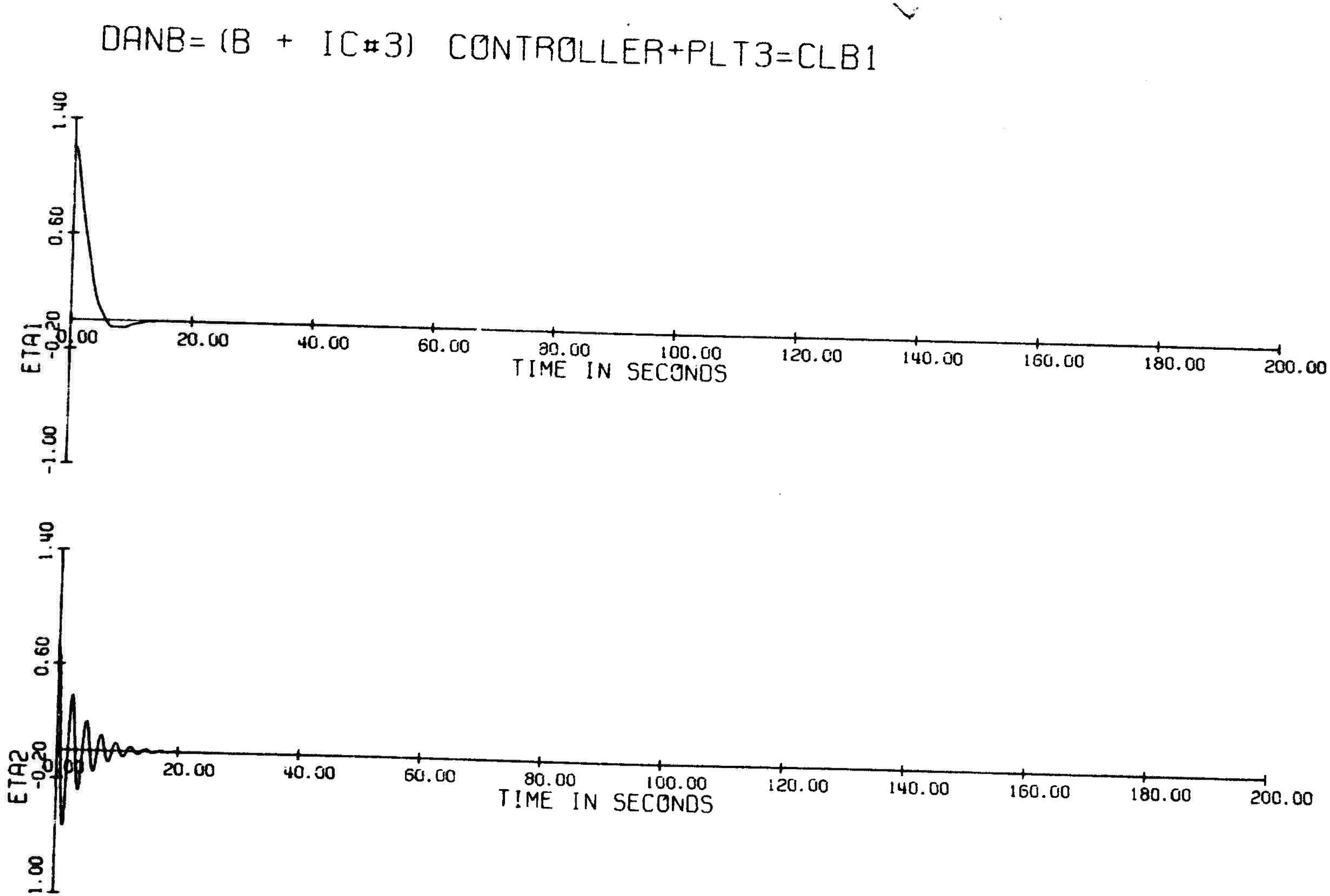

Figure 6-7. **Modal response to initial conditions $q_1(0) = \dot{q}_1(0) = 0$, $q_2(0) = 1$, $\dot{q}_2(0) = 0$ (two sensor configuration).**

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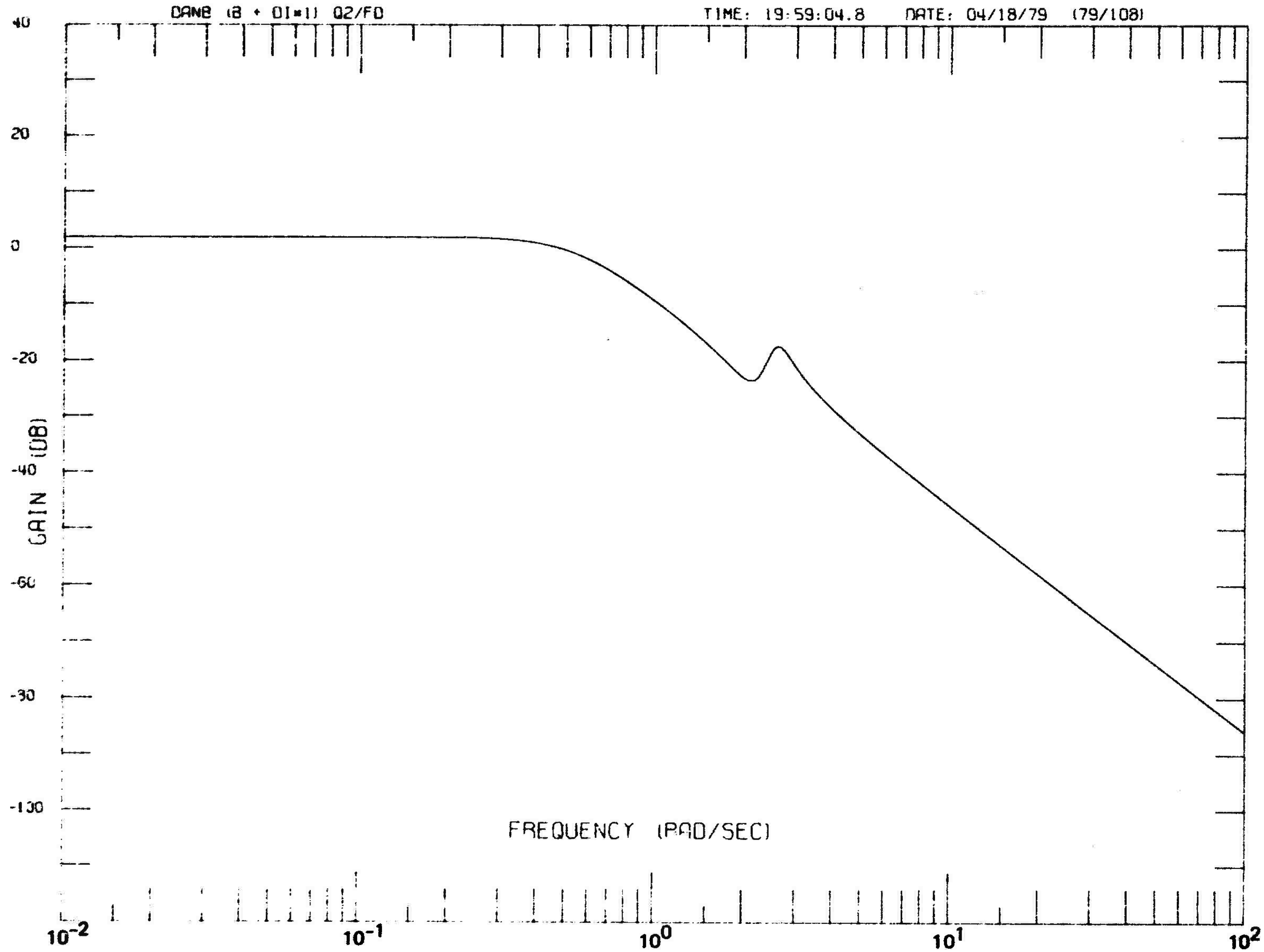

Figure 6-8a. Frequency response (gain) to periodic disturbance $F_2(t) = \sin \omega t$ (two sensor configuration).

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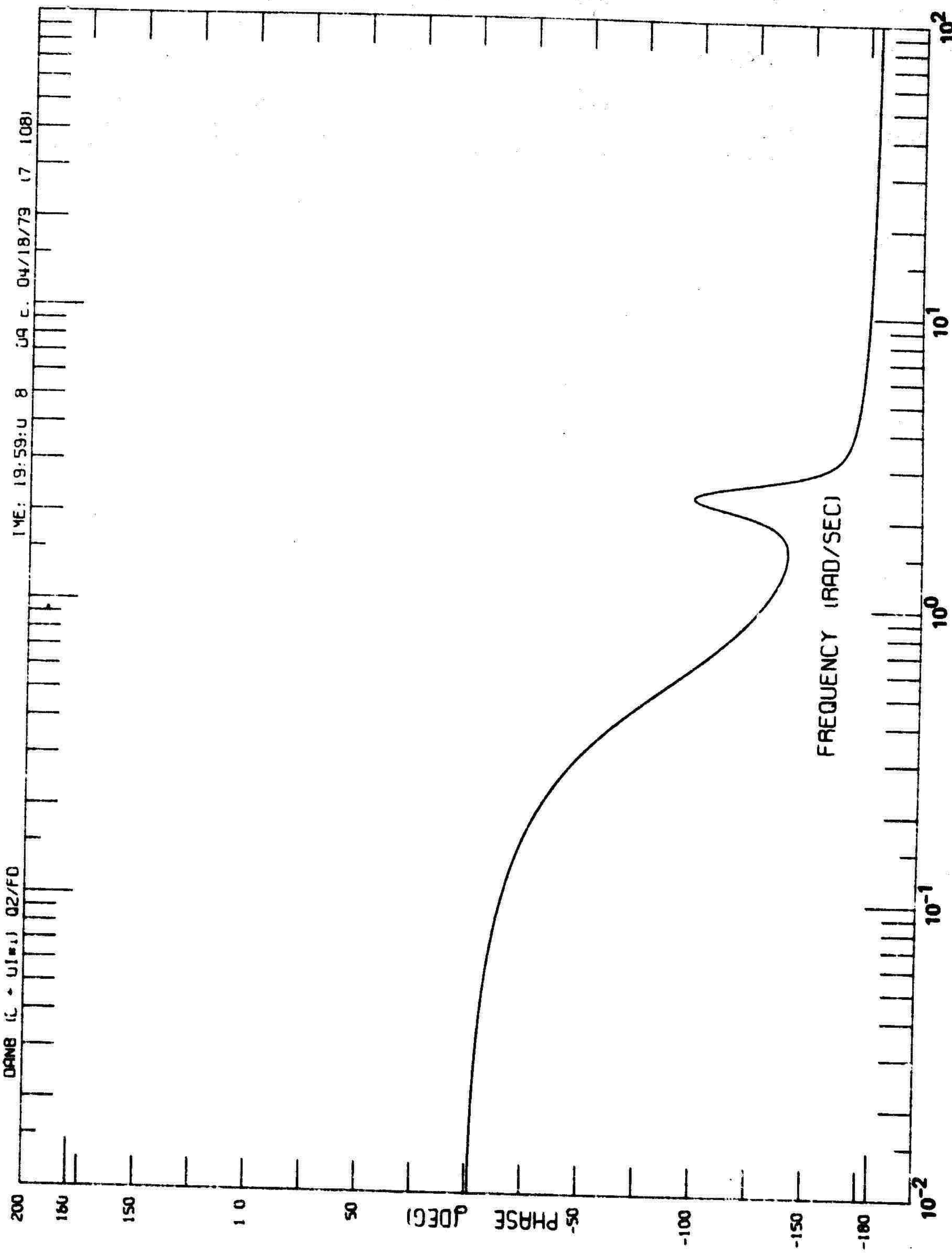

Figure 6-8b. Frequency response (phase) to periodic disturbance $F_2(t) = \sin \omega t$ (two sensor configuration).

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For the specific frequency $\omega=3$ ($\omega_1 < \omega_2 < 3$), plots of the time response in physical coordinates to a periodic disturbance applied to mass 2 are shown in Figure 6-9. Rapid convergence to the steady state condition occurs.

6.3.4 Design Alternatives

The control system designer may wish to consider one or more of the following alternatives within the scope of constant gain output feedback: (1) using a different method of computing the feedback gains, (2) restricting the sensor information available to the control channels, (3) adjusting the modal damping, (4) changing the number of sensors and actuators used, or (5) changing the type of sensors used. Each of these alternatives is briefly discussed.

Without appealing to fundamentally different approaches to constant gain output feedback, such as discussed in other sections of this report, the Kosut method of minimum norm [Section 6.1.2.3] may also be used to compute the gains. A comparison of the two approaches has been made for this example — no significant qualitative or quantitative difference in the overall system dynamic characteristics was observed.

Restricting the information structure by the use of decentralized output feedback [Fig. 6-2] was not investigated for this example. We would expect that this restriction would result in fewer free parameters in the design with the specified sensor configuration.

Adjustment of the amount of damping in both the critical and residual mode is readily accomplished; such adjustment may be in either direction except that the damping ratio in each mode must be positive to ensure stability. Damping for the critical mode is determined by the suboptimal design parameter σ [Eq. (6-47)] whose value can be adjusted by specifying different values for the dynamic parameters ζ^*,ω^* of the optimal reference system [Eq. (6-43)]. Implementation of this change is accomplished by a corresponding selection of the nonzero elements of the state weighting matrix Q [Eq. (6-36), (6-37)]. Damping for the residual mode is determined by the parameter $\varepsilon^{\dagger}$, whose value can be selected as any positive number ζ_{1D} [Eq. (6-53)]. However, modal damping cannot be arbitrarily increased without penalty. Expressions (6-43), (6-48), (6-49), and (6-53) show that increasing the desired values ζ_{1D}, ζ_{2D} for damping in modes 1 or 2, respectively, necessitates increasing the design values for at least one or more of the gain matrix parameters σ, δ, and ε. Since $\psi_1 < 0 < \psi_2$, this results in a monotonic increase in value for the elements of the gain matrix [Eq. (6-45)], which may result in intolerable amplification of unmodeled noise.

† It is assumed here that the constraint relation [Eq. (6-48)] between the free parameters δ and ε is maintained.

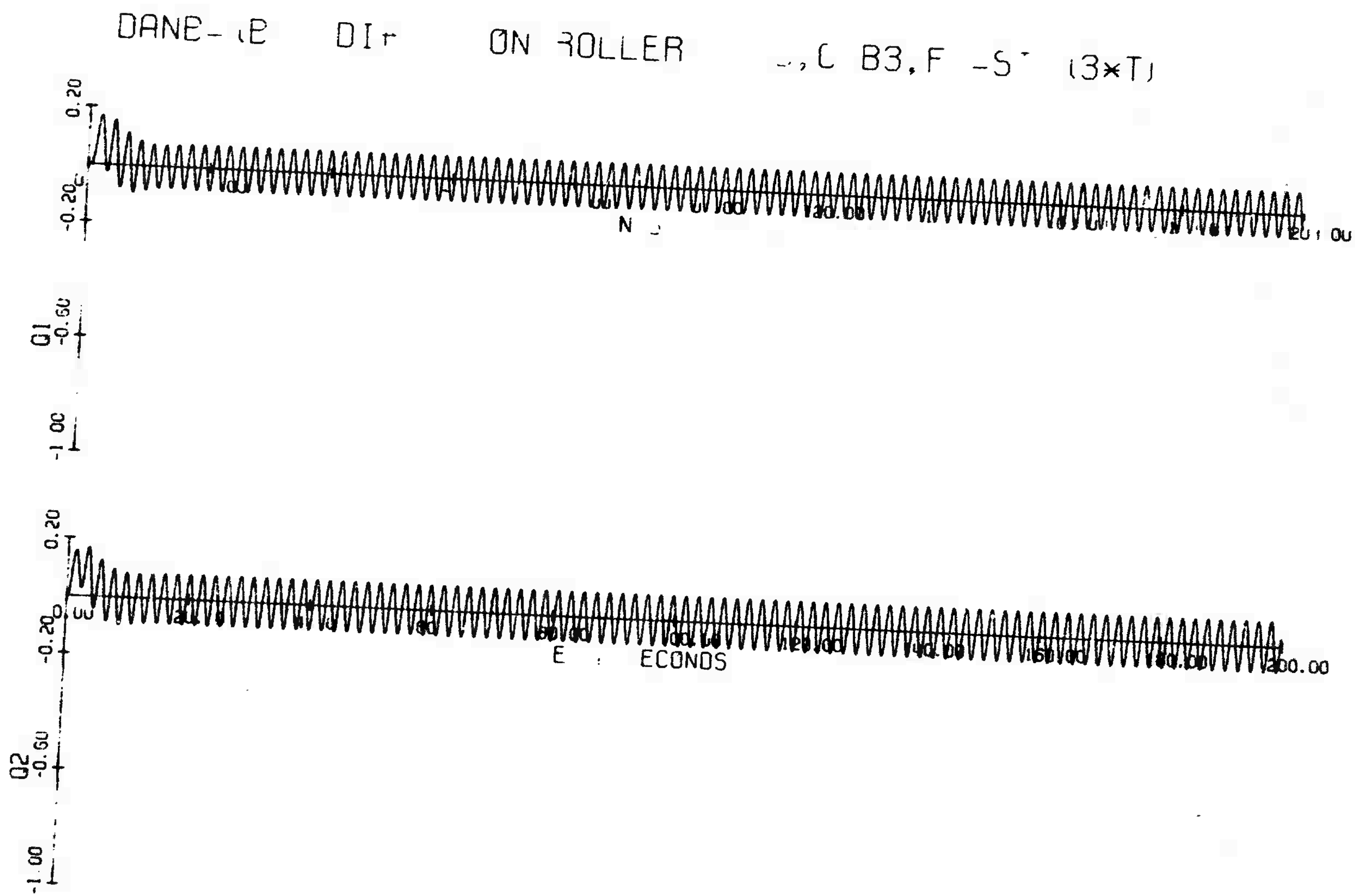

Figure 6-9. Response to periodic disturbance $F_2(t) = \sin 3t$ (two sensor configuration).

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The possibility of reducing the number of actuators has not been investigated to date. However, several important observations have been made regarding reduction in the number of sensors. First, a control design using a single velocity sensor and which stabilizes the full system can be realized only if the sensor is located on the smaller mass (number 1). With the single sensor on the larger mass (number 2), Kosut's methods produce a stable controller for the design model, but the full system cannot be stabilized with any such controller; the source of the instability is observation spillover. Second, the sensor matrix C_c [Eq. (6-25)] for the design model has maximum rank, and hence there is a unique solution for the gain matrix G [Eq. (6-42)]. Therefore, there is no mechanism (free parameters) for accomplishing partial modal decoupling or adjusting residual mode damping as in the two-sensor configuration. The stable single-sensor configuration shows a noticeable performance loss relative to the two-sensor configuration [Figs. 6-10 through 6-15]. This is principally due to the much slower damping of the residual mode (approximately one-tenth of the damping ratio obtainable with the nominal two-sensor design). This effect is strikingly shown in the frequency response plot [Fig. 6-14]: the amplitude peak occurring at the residual mode frequency did not appear in the two-sensor design; moreover, for the single sensor configuration, the phase angle decay in the vicinity of the residual mode frequency is much larger.

Details of the design for each single sensor configuration parallel the development of Section 6.3.2 for the two-sensor configuration. They are briefly outlined in Section 6.5.4.

A design based on the use of position sensors only, instead of velocity sensors, is totally ineffective for the purpose of vibration control. For the sensor configuration consisting of one position sensor only on each mass, there is no mechanism for incorporating damping into a reduced order design model; application of Kosut's methods merely change the natural frequency of the open loop undamped reduced order model. The full system with such a controller is either stable but undamped (all eigenvalues with zero real part) or is unstable. Details of this unsuccessful design are briefly outlined in Section 6.5.4.

6.4 Conclusions

6.4.1 Principal Results

The methods of suboptimal output feedback introduced by Kosut [6] have been carefully evaluated for their potential as design tools for use in developing reduced order controllers for large space structures. Although the work is not regarded as complete [Section 6.4.2], significant progress has been made.

Weaknesses of the Kosut methods especially pertinent to reduced order controller design have been identified [Section 6.2.2]. Most significant are the lack of a stability guarantee, and the inapplicability of the methods when the sensor matrix (in the design model) has a rank deficiency. Theoretical extensions of Kosut's methods have been developed in this report [Section 6.2.3]

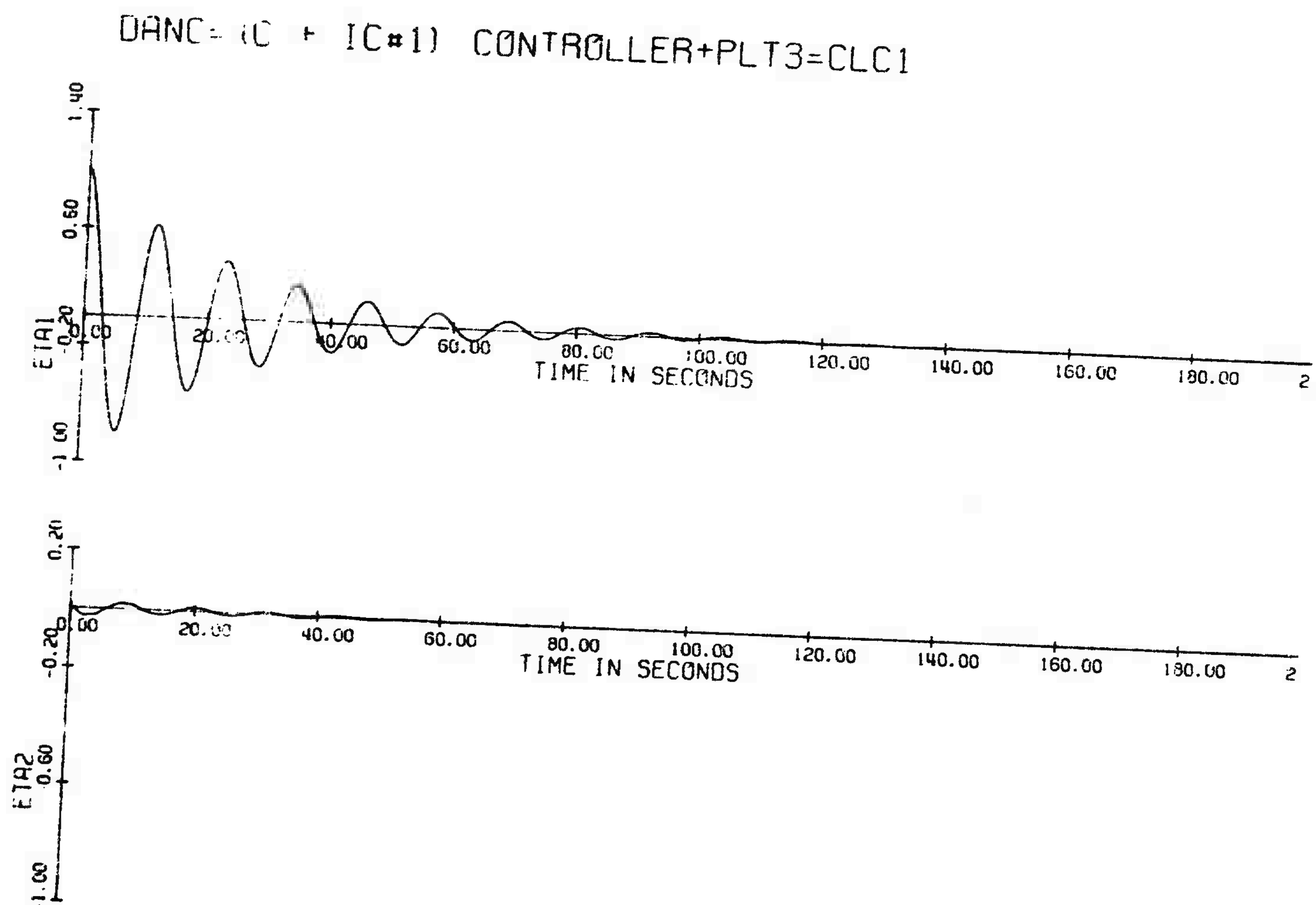

Figure 6-10. Modal response to initial conditions $\eta_1(0) = 1$, $\dot{\eta}_1(0) = 0$, $\eta_2(0) = \dot{\eta}_2(0) = 0$ (stable single sensor configuration).

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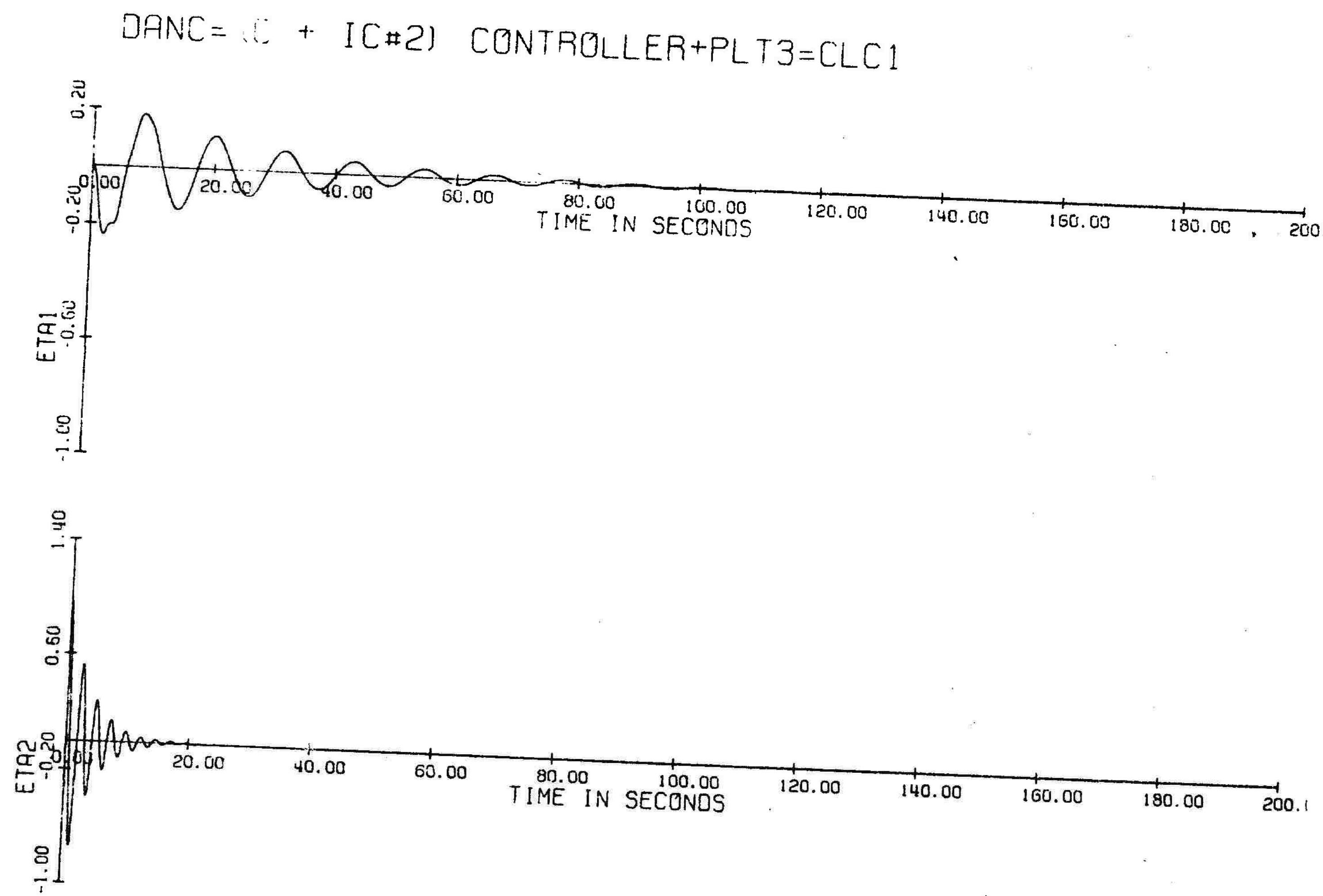

Figure 6-11. Modal response to initial conditions $\eta_1(0) = \dot{\eta}_1(0) = 0$, $\eta_2(0) = 1$, $\dot{\eta}_2(0) = 0$ (stable single sensor configuration).

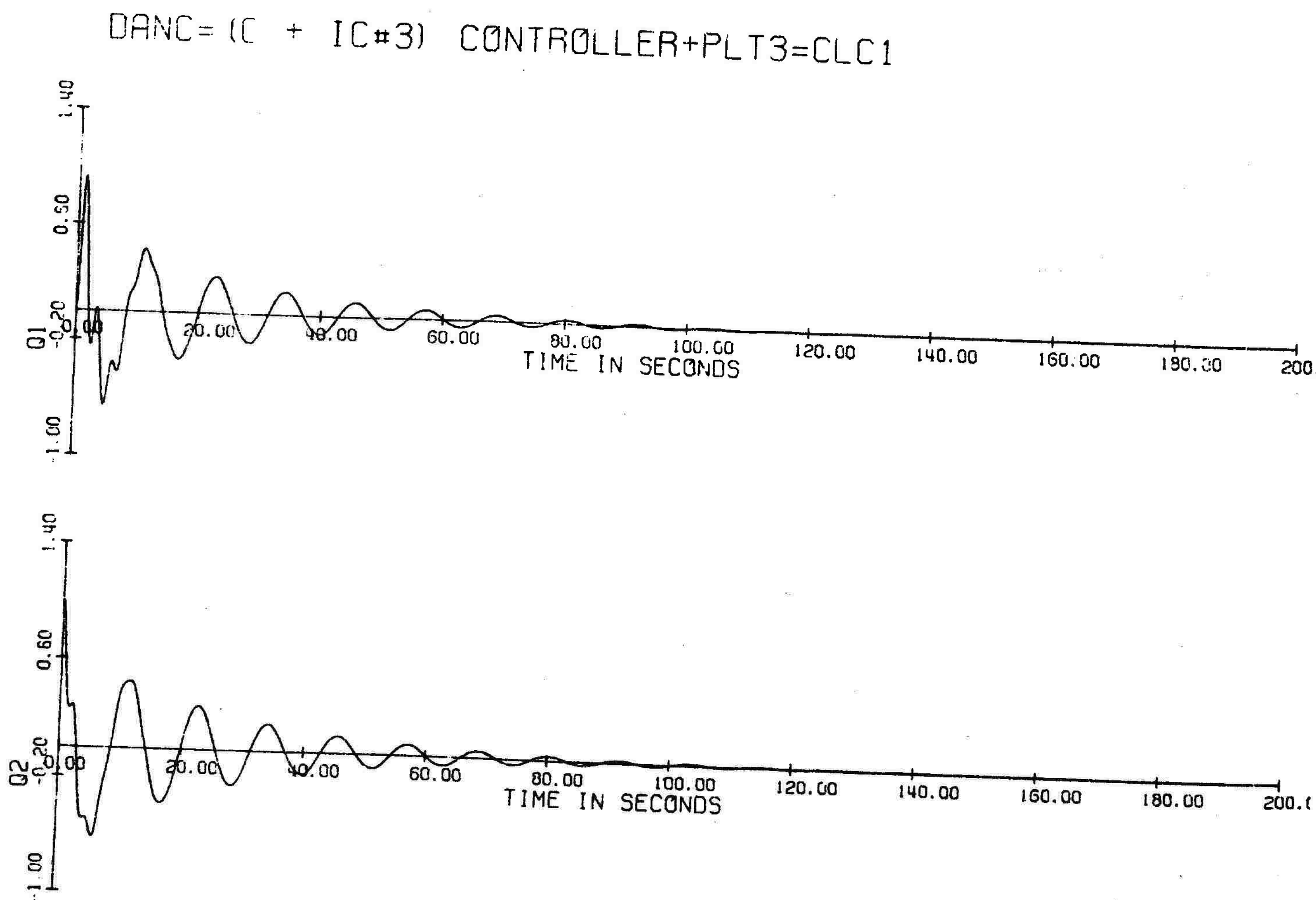

Figure 6-12. Response to initial conditions $q_1(0) = \dot{q}_1(0) = 0$, $q_2(0) = 1$, $\dot{q}_2(0) = 0$ (stable single sensor configuration).

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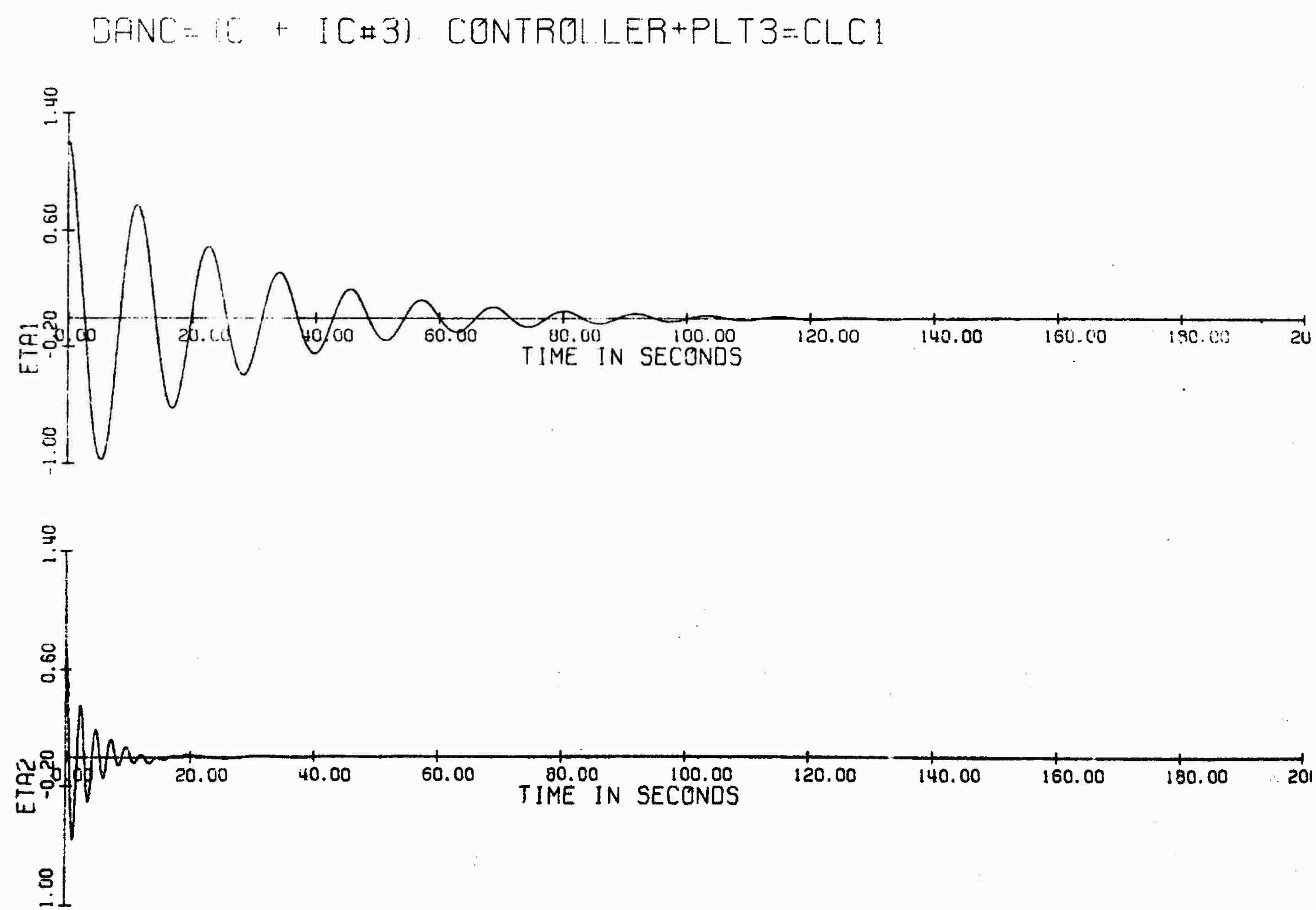

Figure 6-13. Modal response to initial conditions $q_1(0) = \dot{q}_1(0) = 0$, $q_2(0) = 1$, $\dot{q}_2(0) = 0$ (stable single sensor configuration).

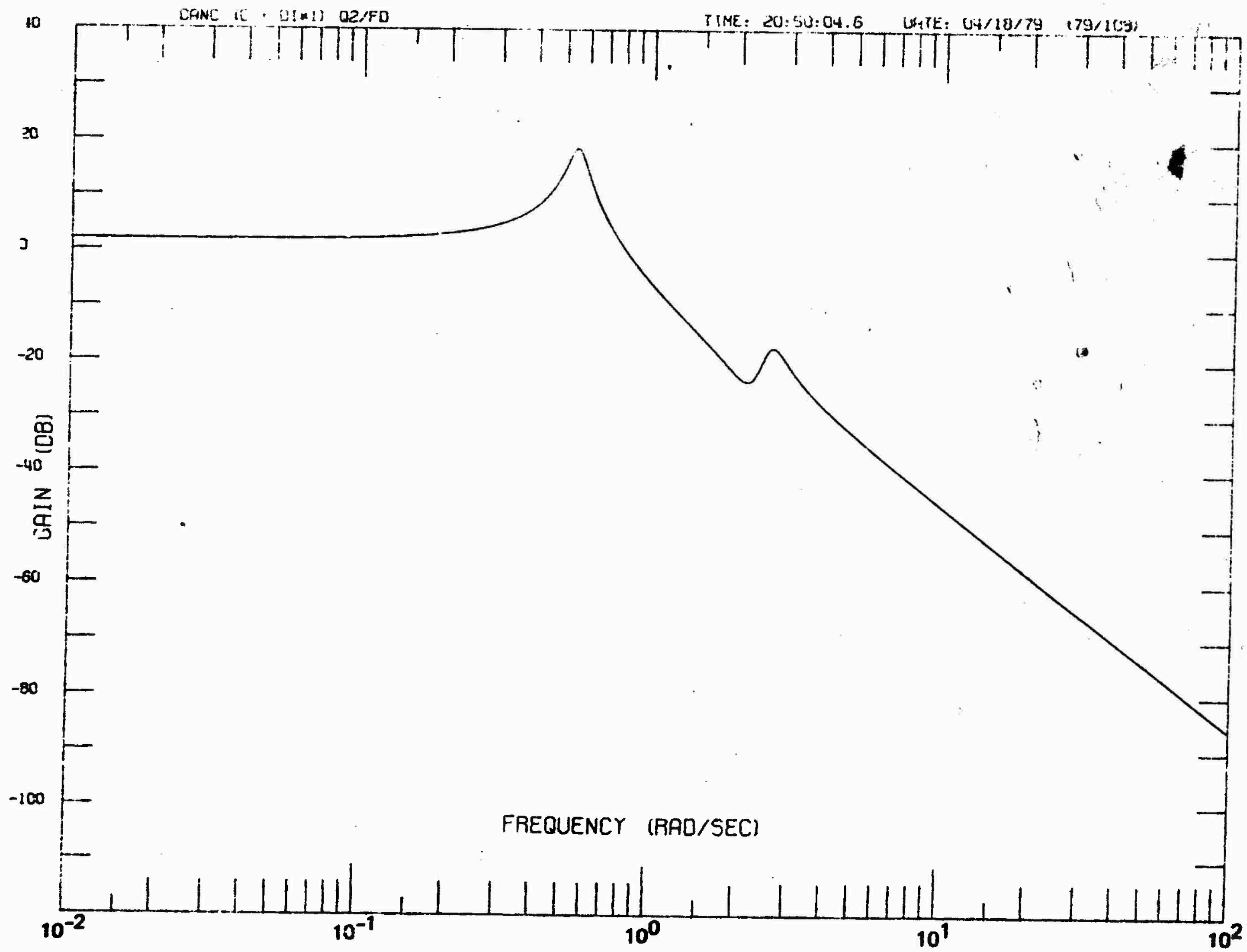

Figure 6-14a. Frequency response (gain) to periodic disturbance $F_2(t)$ = sin ωt (stable single sensor configuration).

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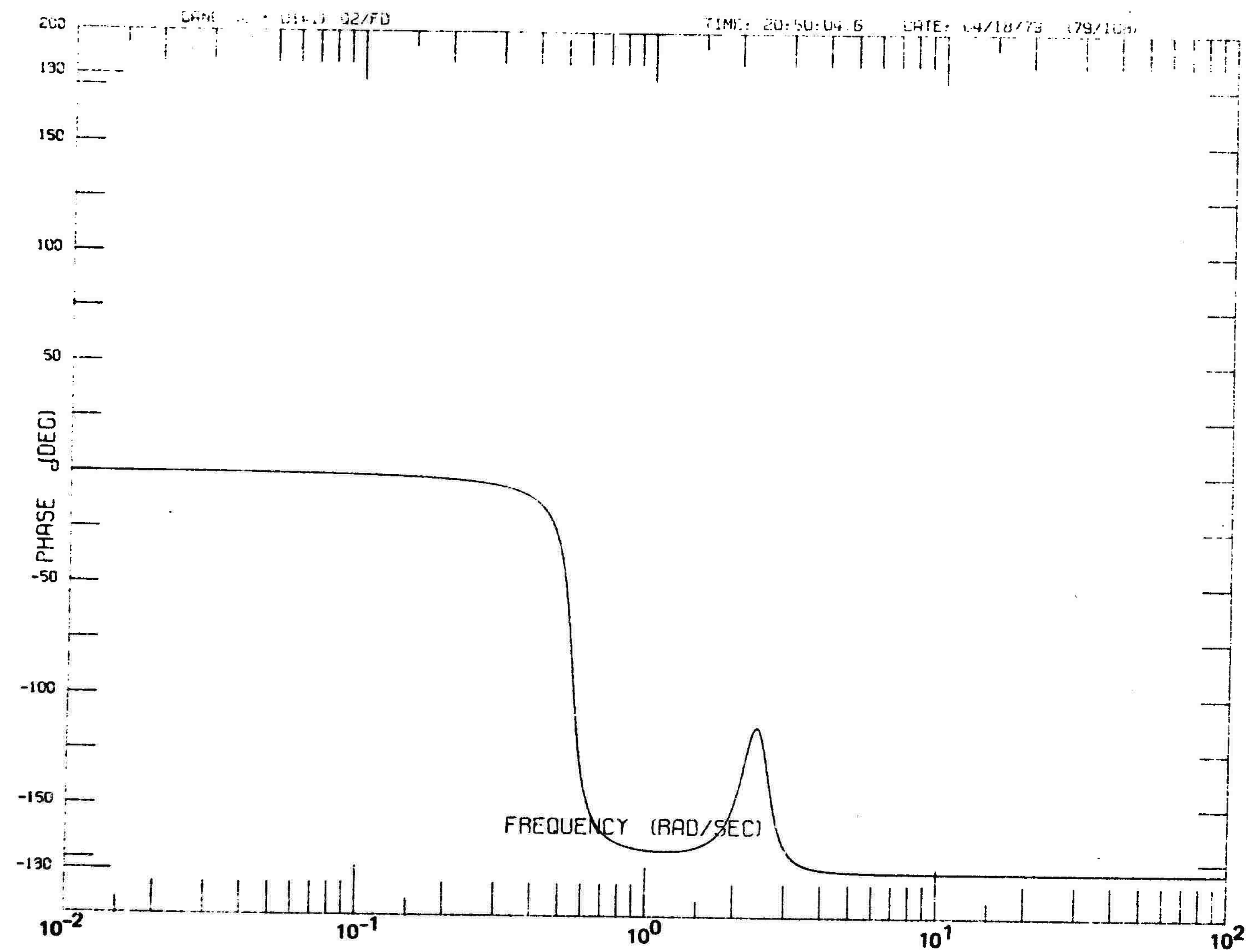

Figure 6-14b. Frequency response (phase) to periodic disturbance $F_2(t) = \sin \omega t$ (stable single sensor configuration)

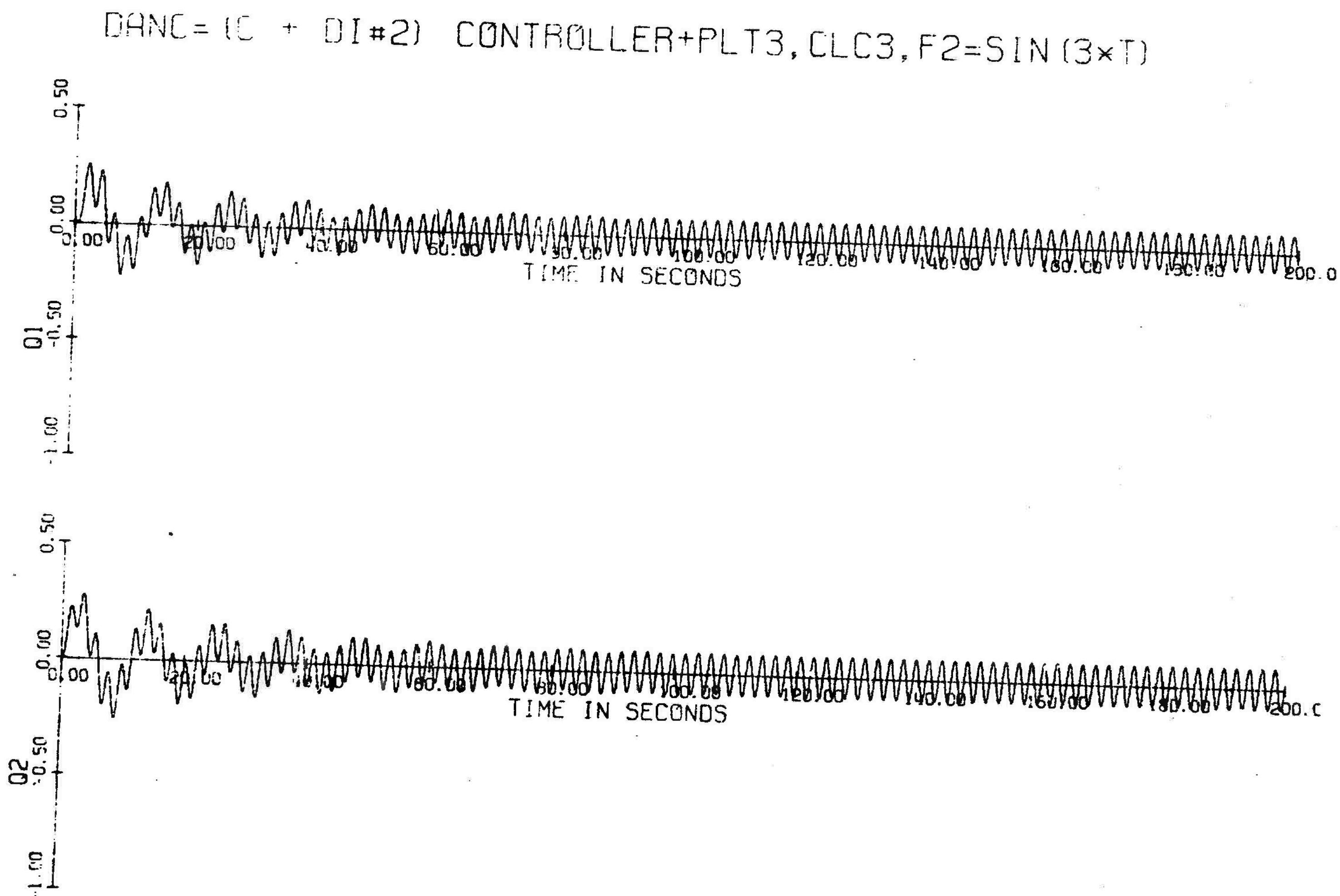

Figure 6-15. Response to periodic disturbance $F_2(t)$ = sin 3t (stable single sensor configuration).

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to address the latter problem; as modified, the Kosut methods can be applied with arbitrary sensor configurations. The implications of this extension are significant, and have been clearly demonstrated in the context of a simple two-mode example [Sections 6.3.3, 6.3.4]. When the extended methods are applied employing sensor matrices with rank deficiency, solution for the feedback gain matrix contains free parameters (proportional in number to the rank deficiency of the sensor matrix). These parameters may be chosen in such a way as to improve the system performance significantly, relative to a corresponding design obtained by adjusting the sensor configuration to eliminate the rank deficiency and then applying the (unextended) Kosut methods. In the example studied, a partial decoupling of the closed loop modal equations was realized. This alleviated the problem of control spillover by eliminating all residual mode excitation of the critical mode. In addition, damping of the residual mode could be adjusted at will. In short, the extension to the Kosut methods developed herein provides a mechanism for using the information available from extra sensors (in addition to the number necessary to ensure stability) to improve the system performance.

6.4.2 Recommendations for Future Work

Initial experience in applying Kosut's design methods, incorporating extensions developed in this report, to a low order vibration control problem has been encouraging. Before making final recommendations regarding the value of these methods for structural vibration control, several additional topics should be treated in future work:

(1) The effects upon control system performance of restricting the information available to each control channel (i.e., of a decentralized output feedback structure) should be determined.

(2) Sensitivity of the control design to modelling errors associated with either lack of fidelity of the evaluation model to the real system, or truncation of the evaluation model to the design model, should be investigated.

(3) The methods should be applied to a more realistic (higher order) structural model.

(4) The methods should be tested against several objective criteria (e.g. [14]) available for determining the degree of suboptimality.

6.5 Appendices

6.5.1 Properties of the Trace Operator

The trace of a square matrix (denoted by Tr $(\cdot)$) is the sum of its diagonal elements. The algebraic properties listed below are found in most standard texts on linear algebra (e.g. [7]). The calculus properties listed, together with many others, have been documented by Athans [11]. All proofs are elementary.

Fact 1: Algebraic Properties

(1) Given $n \times n$ matrices A, B, P with P nonsingular, and a scalar α:

(a) $Tr(A + B) = Tr(A) + Tr(B)$

(b) $Tr(\alpha A) = \alpha Tr(A)$

(c) $Tr(A^T) = Tr(A)$

(d) $Tr(P^{-1}AP) = Tr(A)$

(2) Given bicompatible matrices A, B, with AB and BA square:

$$Tr(BA) = Tr(AB)$$

Fact 2: Differential Calculus Properties

Given matrices A, B and variable matrix X of appropriate sizes:

(1) $\partial/\partial X \; Tr(X) = I$

(2) $\partial/\partial X \; Tr(X^T A) = A$

(3) $\partial/\partial X \; Tr(X^T AXB) = A^T XB^T + AXB$

6.5.2 Suboptimal Design by Modified Minimum Error Excitation

Since the suboptimal system and the optimal reference system satisfy Eq. (6-4), with controllers defined by Eqs. (6-6) and (6-5), respectively, the error vector (6-7) satisfies the relation:

$$\dot{e}(t) = Ae(t) + [BF\ x\ (t) - BF^* x^*(t)] \tag{6-57}$$

To obtain Eq. (6-8), Kosut expressed the bracketed term in Eq. (6-57) in the equivalent form:

$$BFe(t) + B(F-F^*)x^*(t)$$

Alternatively, the bracketed term may be written as

$$B(F-F^*)x(t) + BF^*e(t)$$

so that e(t) satisfies [replacing Eq. (6-8)]

$$\dot{e}(t) = (A+BF^*)e(t) + B(F-F^*)x(t), \quad t \geq 0; \; e(0) = 0 \tag{6-58}$$

The error excitation cost measure [replacing Eq. (6-9)] is

$$I_E'(F) \triangleq \int_0^{+\infty} x^T(t)(F-F^*)^T R(F-F^*)x(t)dt \tag{6-59}$$

Assume A+BF is asymptotically stable. Then integral (6-59) can be reduced to the form of Eq. (6-10) with V satisfying [in place of Eq. (6-11)]

$$(A+BF)^T V + V(A+BF) + (F-F^*)^T R(F-F^*) = 0 \tag{6-60}$$

The suboptimal problem is to minimize the initial-value-free functional

$$\hat{I}_E'(F,V) \triangleq \text{Trace } (V)$$

subject to constraint Eq. (6-60) and control structure constraints on F. Necessary conditions for a solution of this problem, in addition to Eq. (6-60), consist of a multiplier equation [replacing Eq. (6-13)]

$$(A+BF)P + P(A+BF)^T + I = 0 \tag{6-61}$$

and equations for the controller gains. For the case of centralized output feedback (F = GC)

$$G = (F^* - R^{-1}B^T V)[PC^T(CPC^T)^{-1}] \tag{6-62a}$$

For the case of decentralized output feedback ($F_j = g_j^T C_j, j = 1,\ldots,m$)

$$g_j^T = (F_j^* - \frac{1}{r_j} b_j^T V)[PC_j^T(C_j P C_j^T)^{-1}], \; j = 1,\ldots,m \tag{6-62b}$$

where r_j is the j^{th} row element of the diagonal matrix R, and b_j is the j^{th} column vector of B. (These equations replace Eq. (6-14)).

To use these results as a design tool, one attempts to solve Eqs. (6-60) through (6-62) for the matrices V, P, and F. The suboptimal system is stable if and only if the Liapunov Eq. (6-61) has a positive definite solution P [7;p. 254]. Availability of this stability information is due to the inclusion of the suboptimal system trajectories in the cost measure Eq. (6-59).

Unfortunately, there is no general method for finding a closed form solution of Eqs. (6-60) through (6-62) - they must be solved by iteration. In fact, if F* is set to zero, these equations reduce to the Levine-Athans necessary conditions for constant gain optimal output feedback [4].

6.5.3 Proofs of Theoretical Results

The following well-known property of matrices is useful when questions about rank arise. It will be used several times below.

Lemma 1 [10; p. 103] Let A: $\lambda\times\nu$ be a matrix with rank ρ. Then there exist nonsingular matrices S: $\lambda\times\lambda$ and Q: $\nu\times\nu$ such that

$$SAQ = \left[\begin{array}{c|c} I_\rho & 0 \\ \hline 0 & 0 \end{array}\right]$$

where I_ρ is the $\rho\times\rho$ identity matrix. Moreover, S may be chosen so that SA is in row-echelon form.

Since Theorem 6-1 is a special case of Theorem 6-2, only the latter is proved.

Proof of Theorem 6-2. Denote $\rho \triangleq$ rank (C). If $\rho = 0$, both matrices are zero, and hence have the same rank. Otherwise [Lemma 1], there exist nonsingular matrices S: $\lambda\times\lambda$, Q: $\nu\times\nu$ such that $SCQ = \left[\begin{array}{c|c} I_\rho & 0 \\ \hline 0 & 0 \end{array}\right] \triangleq \hat{C}$. Denote $\hat{\pi} \triangleq Q^{-1}\pi(Q^{-1})^T$. Since matrix rank is unchanged by nonsingular matrix multiplication

$$\text{rank}(C\pi C^T) = \text{rank}(S^{-1}\hat{C}\hat{\pi}\hat{C}^T(S^T)^{-1}) = \text{rank}(\hat{C}\hat{\pi}\hat{C}^T)$$

But $\hat{C}\hat{\pi}\hat{C}^T = \left[\begin{array}{c|c} \pi_{11} & 0 \\ \hline 0 & 0 \end{array}\right]$, where π_{11} is the upper left $\rho\times\rho$ submatrix of $\hat{\pi}$. Since π is positive definite, so is $\hat{\pi}$; hence [7; p. 75], det $\pi_{11} > 0$. Thus rank $(\hat{C}\hat{\pi}\hat{C}^T) = \rho$.

Proof of Theorem 6-3. Denote $r \triangleq$ rank (A).

Part (1). If $r = 0$, then $A = 0$, so any matrix is a solution. Otherwise [Lemma 1], there exist nonsingular matrices S: $\lambda\times\lambda$, Q: $\nu\times\nu$ such that $SAQ = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] \triangleq \hat{A}$. Suppose X^0: $\mu\times\nu$ satisfies $X^0A = 0$. Denote $\Gamma \triangleq X^0P^{-1} = [\Gamma_1 \mid \Gamma_2]$, with Γ_1: $\mu\times r$. Then $0 = \Gamma\hat{A} = [\Gamma \mid 0]$, so $\Gamma_1 = 0$ and Γ_2 is arbitrary.

Thus $X^0 = \Gamma S = [0 | \Gamma_2]S$. Conversely, suppose $X^0 = \Gamma S'$ with the structure specified in the theorem; then S' may be used in the reduction of A: $\hat{A} = S'AQ'$. Then $X^0AQ' = \Gamma\hat{A} = [0 | \Gamma']\hat{A} = 0$. Since Q' is nonsingular: $X^0A = 0$.

Part (2). If $r = \lambda$, the matrix Γ of part (1) is the zero matrix: $X^0 = \Gamma S = 0$. Conversely, if $r < \lambda$, part (1) shows that nonzero solutions exist.

The proofs of Theorems 6-4 and 6-5 do not differ significantly from the standard proofs [10] for the case that X is a row vector; they are therefore omitted.

The proof of Theorem 6-6 is facilitated by the following lemma.

Lemma 2. If $\rho \varepsilon \{1,\ldots,\lambda\}$ such that $\operatorname{rank}(C) < \rho$, then $\operatorname{rank}\left[\frac{C\pi C^T}{F^*\pi C^T}\right] < \rho$

Proof of Lemma 2. If $\rho = 1$, then both matrices are zero and so each has rank $0 < \rho$. Otherwise, denote $E \triangleq \left[\frac{C}{F^*}\right]\pi$, and consider the product EC^T. Choose integers $1 \leq i_1 <\ldots< i_\rho \leq \lambda + \mu$, $1 \leq j_1 <\ldots< j_\rho \leq \lambda$, and form the $\rho\times\rho$ submatrix $EC^T[i_1,\ldots,i_\rho | j_1,\ldots,j_\rho]$, consisting of rows $i_1,\ldots,i_\rho$ and columns $j_1,\ldots,j_\rho$ of EC^T. With the same notation, note that this can be represented as a product of submatrices:

$$EC^T[i_1,\ldots,i_\rho | j_1,\ldots,j_\rho] = E[i_1,\ldots,i_\rho | 1,\ldots,\nu]C^T[1,\ldots,n | j_1,\ldots j_\rho]$$

If $\rho > \nu$, then $\operatorname{rank} EC^T[i_1,\ldots,i_\rho | j_1,\ldots,j_\rho] \leq \nu < \rho$, so its determinant vanishes. Otherwise, using the determinant expansion for products of rectangular matrices [10; p. 127]:

$$\det EC^T[i_1,\ldots,i_\rho | j_1,\ldots,j_\rho] = \sum_{1 \leq \ell_1 <\ldots<\ell_\rho \leq \nu} \det E[i_1,\ldots,i_\rho | \ell_1,\ldots,\ell_\rho]\cdot$$

$$\det C^T\left[\ell_1,\ldots,\ell_\rho | j_1,\ldots,j_\rho\right]$$

Since $\operatorname{rank}(C) < \rho$, each of the $\rho\times\rho$ determinants in the sum derived from C^T vanishes. Hence, each $\rho\times\rho$ minor of EC^T vanishes, and so $\operatorname{rank}(EC^T) < \rho$.

Proof of Theorem 6-6. Use the symbol E defined in the proof of Lemma 2 above.

($\geq$ part). The rank of a matrix is equal to the number of its linearly independent rows. Hence, $\operatorname{rank}(C\pi C^T) \leq \operatorname{rank}(EC^T)$.

($\leq$ part). Suppose rank $(C\pi C^T) <$ rank(EC^T). Set $\rho \triangleq$ rank(EC^T). Using Theorem 6-2, it follows that rank$(C) < \rho$. Then Lemma 2 implies that rank$(EC^T) < \rho$, which contradicts the definition of ρ. Hence, rank$(EC^T) \leq$ rank$(C\pi C^T)$.

6.5.4 Suboptimal Design for Alternate Sensor Configurations

The multiplier matrix P associated with the method of minimum error excitation is determined by the optimal reference system [Eq. (6-13)] independent of the sensor configuration. Hence, each of the subsections to follow use the same matrix P as computed for the nominal sensor configuration [Eq. (6-41)].

6.5.4.1 Configuration I: Single Velocity Sensor (Mass 1)

The sensor matrix $\hat{C}$ [Eq. (6-25)] is

$$\hat{C} = [0\ \psi_1 | 0\ \phi_1] \equiv [C_c | C_R];$$

hence the matrix C_c used in the control design has maximum rank. However, in this case, $C_c P C_c^T$ is invertible, so Eq. (6-42) has the unique solution

$$G = F^* P C_c^T (C_c P C_c^T)^{-1} = \begin{bmatrix} -\sigma \\ \\ -\sigma \dfrac{\psi_2}{\psi_1} \end{bmatrix} \tag{6-63}$$

where σ is as defined by Eq. (6-43). The overall gain matrix $F \triangleq GC_c$ is identical with Eq. (6-46), hence the closed loop design model has the same dynamic characteristics as the corresponding design model for the two-sensor configuration ($\omega = \omega_2$; $\zeta = 0.1$).

The characteristic equation of the full system incorporating the controller generated by Eq. (6-63) cannot be factored analytically

$$\det\,(A+B[G\hat{C}] - \lambda I)$$

$$= [\lambda^2 + \sigma(\psi^T\psi)\lambda + \omega_2^2][\lambda^2 + \sigma(\phi^T\psi)\frac{\phi_1}{\psi_1}\lambda + \omega_1^2] - \sigma^2(\phi^T\psi)(\psi^T\psi)\,\frac{\phi_1}{\psi_1}\,\lambda^2$$

However, it is easily shown using the Routh-Hurwitz criteria that the full system is stable for all $\sigma > 0$. The dynamic characteristics of the system evaluated for the design value of σ [Eq. (6-43)] are:

Mode 2 (critical): $\omega_{2c} = 2.5803842$ ($=0.9967\ \omega_2$)

$\zeta_{2c} = 0.10024986$ ($=1.0025\ \zeta_{2D}$)

Mode 1 (residual): $\omega_{1c} = 0.54806316$ ($=1.003\ \omega_1$)

$\zeta_{1c} = 0.072724483$ ($=0.1028\ \zeta_{1D} \doteq 1/10\ \zeta_{1D}$)

6.5.4.2 Configuration II: Single Velocity Sensor (Mass 2)

The sensor matrix $\hat{C}$ [Eq. (6-25)] is

$$\hat{C} = [0\ \psi_2 | 0\ \phi_2] \equiv [C_c \vdots C_R]$$

so C_c has maximum rank. Hence, Eq. (6-42) has a unique solution

$$G = F^* P C_c^T (C_c P C_c^T)^{-1} = \begin{bmatrix} -\sigma \dfrac{\psi_1}{\psi_2} \\ \\ -\sigma \end{bmatrix} \tag{6-64}$$

where σ is as defined in Eq. (6-43). The gain matrix $F \triangleq GC_c$ is identical with Eq. (6-46); hence the closed loop design model is stable with the same dynamic characteristics as the corresponding design model for the two sensor configuration ($\omega=\omega_2$; $\zeta=0.1$).

The characteristic equation for the full system incorporating the controller generated by Eq. (6-64) is

$$\det\,(A+B[G\hat{C}] - \lambda I) \tag{6-65}$$

$$= [\lambda^2 + \sigma(\psi^T\psi)\lambda + \omega_2^2][\lambda^2 + \sigma(\phi^T\psi)\frac{\phi_2}{\psi_2}\lambda + \omega_1^2] - \sigma^2(\phi^T\psi)(\psi^T\psi)\frac{\phi_2}{\psi_2}\lambda^2$$

Expansion of Eq. (6-65) in powers of λ reveals that the coefficients of λ^3 and λ^1 have coefficients of the form $\alpha_3\sigma$ and $\alpha_1\sigma$, respectively, where α_1 and α_3 are constants (depending on system parameters) satisfying $\alpha_1 < 0 < \alpha_3$. Hence it is not possible to stabilize the system with this design method for any real σ. The dynamic characteristics of the system evaluated for the design value of σ [Eq. (6-43)] are:

Mode 2 (critical): $\omega_{2c} = 2.6109331$ ($\approx 1.0086\ \omega_2$)

$\zeta_{2c} = 0.098330464$ ($= 0.9833\ \zeta_{2D}$)

Mode 1 (residual): $\omega_{1c} = 0.54165065$ ($\approx 0.9915\ \omega_1$)

$\zeta_{1c} = -0.19840072 < 0$

The fundamental source of this instability is observation spillover. This is seen by writing the differential equation for mode 1 in the full system

$$\ddot{\eta}_1 + \sigma(\phi^T\psi)\frac{\phi_2}{\psi_2}\dot{\eta}_1 + \omega_1^2\eta_1 = -\sigma(\phi^T\psi)\dot{\eta}_2$$

The coefficient of $\dot{\eta}_1$ is negative for all $\sigma > 0$; this coefficient drives the instability and is introduced into the equation by taking into account the contribution to the sensor measurements (ignored in the reduced order design) due to excitation of the residual mode.

6.5.4.3 Configuration III: One Position Sensor on Each Mass

The sensor matrix $\hat{C}$ [Eq. (6-25)] is

$$\hat{C} = \left[\begin{array}{cc|cc} \psi_1 & 0 & \phi_1 & 0 \\ \psi_2 & 0 & \phi_2 & 0 \end{array}\right] \equiv [C_c \mid C_R]$$

so C_c has a rank deficiency of 1; hence [Theorem 6-2], rank $(C_c P C_c^T) = 1$. The gain Eq. (6-42) has multiple solutions [Theorem 6-6] of the form

$$G(\varepsilon,\delta) = \begin{bmatrix} -\tau + \varepsilon\frac{\psi_2}{\psi_1} & -\varepsilon \\ -\delta & -\tau + \delta\frac{\psi_1}{\psi_2} \end{bmatrix} \qquad (6\text{-}66)$$

where ε and δ are arbitrary parameters, and τ is given by [cf. Eq. (6-43)]

$$\tau = \frac{k_{12}p_{11}+k_{22}p_{12}}{p_{11}} = \frac{1}{(\psi^T\psi)}\left[\frac{(\omega^*)^2[1+(\omega^*)^2]}{1+(\omega^*)^2[1+4(\zeta^*)^2]} - \omega_2^2\right] = -0.18729075$$

The closed loop design model has the overall gain matrix $F = GC_c$

$$F = \begin{bmatrix} -\tau\psi_1 & 0 \\ -\tau\psi_2 & 0 \end{bmatrix}$$

independent of the free parameters, ε and δ. The characteristic polynomial $\det(A_c+B_cF-\lambda I)$ has the form

$$\lambda^2 + [\omega_2^2 + \tau(\psi^T\psi)]$$

hence the design model is undamped with a natural frequency slightly less than the critical mode natural frequency.

The characteristic polynomial of the full system incorporating the controller generated by Eq. (6-66) is

$$\det(A+B[C\hat{C}]-\lambda I) =$$

$$[\lambda^2+\omega_2^2+\tau(\psi^T\psi)]\;[\lambda^2+\omega_1^2+\tau(\phi^T\phi)+(\delta\frac{\phi_2}{\psi_2}-\varepsilon\frac{\phi_1}{\psi_1})\det\Phi] - \tau(\phi^T\psi)\;[\tau(\phi^T\psi)+(\delta-\varepsilon)\det\Phi]$$

which is quadratic in λ^2. Hence, roots of the fourth order polynomial may be written in the form

$$\lambda = \sqrt{\rho_1}e^{i\theta_1/2},\ \sqrt{\rho_1}e^{i(\theta_1/2+\pi)};\ \sqrt{\rho_2}e^{i\theta_2/2},\ \sqrt{\rho_2}e^{i(\theta_2/2+\pi)}$$

with $\rho_j \geq 0$, $0 \leq \theta_j < 2\pi$, $j = 1,2$. It follows that either all roots lie on the imaginary axis, or at least one root lies in the right half plane. Hence the full system cannot be made asymptotically stable, regardless of how the design parameters τ, ε, and δ are chosen.

References

1. Athans, M. and Falb, P.L., Optimal Control. New York: McGraw-Hill, 1966.

2. Luenberger, D.G., An Introduction to Observers, IEEE Trans. Automatic Control, Vol. AC-16, pp. 596-602, December 1971.

3. Kalman, R.E. and Bucy, R.S., New Results in Linear Filtering and Prediction Theory, Trans. ASME, J. Basic Engineering, Vol. 83, pp. 95-108, March 1961.

4. Levine, W.S., Johnson, T.L. and Athans, M., Optimal Limited State Variable Feedback Controllers for Linear Systems, IEEE Trans. Automatic Control, Vol. AC-16, pp. 785-793, December 1971.

5. Bingulac, S.P., Cuk, N.M. and Calovic, M.S., Calculation of Optimum Feedback Gains for Output-Constrained Regulators, IEEE Trans. Automatic Control, Vol. AC-20, pp. 164-166, February 1975.

6. Kosut, R.L., Suboptimal Control of Linear Time-Invariant Systems Subject to Control Structure Constraints, IEEE Trans. Automatic Control, Vol. AC-15, pp. 557-563, October 1970.

7. Bellman, R., Introduction to Matrix Analysis, New York: McGraw-Hill, 1970, 2nd ed.

8. Petkovski, Dj.B. and Rakic, M., On the Calculation of Optimum Feedback Gains for Output-Constrained Regulators, IEEE Trans. Automatic Control, Vol. AC-23, p. 760, August 1978.

9. Gantmacher, F.R., The Theory of Matrices, Vol. I. New York: Chelsea, 1960.

10. Thrall, R.M. and Tornheim, L., Vector Spaces and Matrices, New York: Wiley, 1957.

11. Athans, M., The Matrix Minimum Principle, Information and Control, Vol. 11, pp. 592-606, November-December, 1967.

12. Rynaski, E.G., Reynolds, P.A., and Shed, W.H., Design of Linear Flight Control Systems Using Optimal Control Theory, USAF Flight Dynamics Lab., Rept. ASD-TDR-63-376, April 1964.

13. Jacobs, O.L.R., The Damping Ratio of an Optimal System, IEEE Trans. Automatic Control, Vol. AC-10, pp. 473-476, October 1965.

14. McClamroch, N.H., Evaluation of Suboptimality and Sensitivity in Control and Filtering Processes, IEEE Trans. Automatic Control, Vol. AC-14, pp. 282-285, June 1969.

SECTION 7

STOCHASTIC OPTIMAL OUTPUT FEEDBACK CONTROL

7.1 Background

7.1.1 Introduction: Main Ideas and Underlying Theory

The flexure dynamics of a large space structure in orbit are approximated by a high-order system of linear differential equations, which are excited primarily by force and torque perturbations due to the operation of equipment on the structure. The problem of using feedback controls to actively reduce the line-of-sight errors induced by these disturbances is considered here. It is assumed that a finite number of sensors and actuators have been placed on the structure and that the feedback compensation is constrained to be linear and of zero dynamic order, i.e., that each actuator input signal is to be synthesized directly as a linear combination of the sensor output signals at each instant of time.

The stochastic output feedback control (SOFC) method is one approach proposed for the determination of output-feedback gains. The basic idea is to approximate the disturbances by stationary second-order random processes; the statistics of the deflections and deflection rates for any set of feedback gains can then be derived. Significant responses such as stresses, accelerations, critical vibration mode deflections and rates, and particularly line-of-sight errors are expressed as linear combinations of the deflections and deflection rates; thus the response statistics can be computed. The "best" set of gains is taken to be that which minimizes a nonnegative linear combination of the mean-square responses. The principal design parameters are the relative weightings given to the significant responses. The key technical contribution of the method described here is a procedure for computing the best gains.

The potential advantages of the stochastic output feedback control problem formulation are:

(1) Relatively few design parameters; freedom from unpredictable errors in designer judgment which can occur in pole-placement approaches.

(2) Robustness (e.g., gain and phase margins) of the optimal gains, which is due to the stochastic conceptualization of the problem.

(3) Ability to make a tradeoff between fast regulation on the one hand, and sensitivity to disturbances and the effects of unmodelled residual modes on the other.

The last point is particularly worth noting: an increase in feedback gains, generally, will tend to reduce mean-square responses but at the same time will use more control energy and make the system more sensitive to sensor noise and residual mode effects—a stochastic formulation of the problem incorporates the tradeoff between these favorable and unfavorable effects, yielding optimal gains which are bounded.

While the method is in many respects a culmination of the state-of-the-art in optimal multivariable compensator design theory, and hence benefits from a good deal of "vicarious experience," the detailed properties of the specific algorithms presented herein have only been explored in the context of the two-mode example reported in Section 7.3. The results of this example appear to be promising.

The key ideas for this approach to stochastic control of infinite-dimensional systems have been described in Reference 1. The conceptual background for the particular problem formulation reported here may be traced to the mean-square error methods for single-input, single-output plants developed in the early text by Newton, Gould, and Kaiser [2], and in the early paper by Axsäter [3]. The subsequent development of ideas may be followed through a sequence of papers by Levine, Johnson and Athans [4], Platzman and Johnson [5], Blanvillain and Johnson [6,7], and Naije and Bosgra [8], as well as references cited therein.

In the remainder of Section 7.1, the technical highlights of the SOFC approach are outlined and the assumptions are summarized. In Section 7.2, the relative strengths and weaknesses are compared with other methods; Section 7.3 is devoted to the example. Section 7.4 contains an overview, summary and conclusions.

7.1.2 Outline of the Design Method and Algorithm

A solution of the problem is developed in three phases:

(1) A number of synthetic sensor and actuator signals, each equal to the number of critical modes, is formed by taking linear combinations of the physical sensor outputs and actuator inputs which are available.

(2) The magnitudes of the open-loop effects of the residual mode signals on the critical mode dynamics and synthetic sensor outputs are estimated, and the estimates are used to determine the magnitudes of statistically-equivalent random disturbances and sensor errors (in addition to the direct effects of physical disturbances and inherent sensor noise). The SOFC problem for a truncated model of the structure containing only critical mode dynamics can then be stated.

(3) The best feedback gains for the truncated model are determined using a recently-developed algorithm for solving the SOFC problem. Stability, in general, must be verified through a simulation involving the residual modes which are available.

Phases (2) and (3) may be iterated, if necessary, to improve closed-loop properties; the relative weightings on the mean-square response errors can also be adjusted during the design procedure.

7.1.2.1 Statement of the General Problem

A very high-order finite-element model of the linearized flexure dynamics is assumed to be available in the form

$$M\ddot{q} + Kq = F \tag{7-1}$$

where

q = vector of physical deflection or hybrid-deflection variables.

F = vector of total external forces or torques, including control and disturbance inputs.

M = generalized mass matrix.

K = generalized stiffness matrix.

It is assumed that static deformations and forces have been eliminated in defining the variables in Eq. (7-1). The model (7-1) is also assumed to be sufficiently refined that the effect of still higher order dynamics is truly insignificant, and that these modes may thus be safely ignored.* Through an eigenvalue analysis, a solution of

$$K\Phi = M\Phi\Omega^2; \; \Phi^T M\Phi = I$$

where $\Omega^2 = \text{diag}\,[\omega_1^2, \ldots, \omega_n^2]$ and $I = \text{diag}\,[1, 1, \ldots, 1]$ is assumed to be available. Through definition of the modal coordinates

$$q = \Phi\eta$$

the system is brought into modal form

$$\ddot{\eta} + \Omega^2\eta = \Phi^T F \tag{7-2}$$

In Ω^2, suppose that the modes have been ordered so that

$$\Omega^2 = \begin{bmatrix} \Omega_c^2 & 0 \\ 0 & \Omega_r^2 \end{bmatrix}$$

where the number of critical modes, n_c, (dimension of Ω_c^2), is of the order that can be retained for purposes of design and/or numerical computations. Typical values might be $n \sim 1000$, $n_c \sim 30$.

Furthermore, suppose that in (7-2)

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*See, however, suggestions for further research.

$$F = F_a + F_e = B_a u + F_e$$

where F_a are actuator forces (linearly related to the control input signals, u), and F_e represents external disturbance forces. The measurements are assumed to be

$$y = C_s \dot{q} + e$$

where e represents inherent physical measurement noise.

In this analysis, it is assumed that only deflection rates can be measured; the more general case can also be carried through. It is further assumed that u and y both have dimensions greater than n_c; again, this stipulation can be removed.

The control problem is to find a set of output-feedback gains, G, such that for

$$u = -Gy$$

the closed-loop system

$$\ddot{\eta} + \Phi^T B_a G C_s \Phi \dot{\eta} + \Omega^2 \eta = \Phi^T F_e - \Phi^T B_a G e \qquad (7\text{-}3)$$

is asymptotically stable. By partitioning η into critical and residual modes, as $\eta = \begin{bmatrix} \eta_c \\ \hline \eta_r \end{bmatrix}$, it is readily apparent from Eq. (7-3) that feedback in general will produce coupling between critical and residual modes, which has been termed "control and observation" spillover. Define the damping matrix to be $D = \begin{bmatrix} D_{cc} & D_{cr} \\ D_{rc} & D_{rr} \end{bmatrix} = \Phi^T B_a G C_s \Phi$. The responses of the system, in general, can be approximated by linear combinations of the mode deflections and deflection rates

$$r = H\eta + L\dot{\eta} \qquad (7\text{-}4)$$

For example, r might represent (linearized) line-of-sight error, or critical mode deflections.

To apply the stochastic output feedback control approach, F_e and e are approximated as stationary white noise processes; then for any stabilizing set of gains, G, $\eta(t)$ becomes a stationary second-order process as $t \to \infty$, and r(t) also becomes a random process. Assuming that stabilizing gains exist, we seek an optimum set of gains, G*, which minimizes the weighted asymptotic covariance of the response, denoted by

$$J(G) = \lim_{t \to \infty} E \left\{ r^T(t)\, Q_r r(t) \right\} \tag{7-5}$$

where Q_r is a positive definite design matrix which can be used to represent the relative importance of different responses. For instance, if $Q_r = I$, $H = [I \vdots 0]$, and $L = [0 \vdots 0]$, then G will be chosen to minimize the sum of the mean-square critical mode deflections.

The algorithm presented in Section 7.1.2.4 can be used to address this large-scale problem directly, but it is of much greater practical interest to impose a constraint that only critical mode dynamics be employed in the final design calculations. Prior knowledge of residual mode shapes and frequencies is assumed to be available for purposes of model truncation, however. The next two subsections are devoted to the treatment of this important constraint.

7.1.2.2 Model Truncation

Only a limited number of variables can be retained for purposes of control design; in this case, a subset of the open-loop modes, termed the critical modes, is retained. We shall describe a procedure for truncating the remaining (residual) modes from the design model. Truncation can be viewed as a form of model aggregation; while it is not necessarily optimum, it has been selected here because it is practical and easy to comprehend. It is reasonable to assume, in doing so, that the (key) responses are correspondingly truncated to include only critical mode components (in effect, this constitutes one definition of a critical mode); beyond stabilizing the residual modes so that the limit in Eq. (7-5) exists, all interest in detailed control of the residual modes is thus given up.

The proposed method for achieving this is through combined sensor and actuator signals. Let

$$u = [A_c]\, \bar{u} \qquad \bar{y} = [S_c] y \tag{7-6}$$

where $\bar{u}$, $\bar{y}$ are of dimension n_c (recall that u and y had dimensions of say, n_a, n_s, greater than n_c, by assumption). Furthermore, suppose that B_a and C_s are conformably partitioned as

$$B_a = \begin{bmatrix} B_c \\ B_r \end{bmatrix} \quad C_s = [C_c C_r]$$

Truncated control-gains M_c will be in the form of a feedback from $\bar{u}$ to $\bar{y}$

$$\bar{u} = M_c \bar{y}$$

Define the transformed actuator and sensor matrices as

$$\tilde{B} = \Phi^T B_a = \begin{bmatrix} \tilde{B}_c \\ \tilde{B}_r \end{bmatrix} \text{ and } \tilde{C} = C_s \Phi = [\tilde{C}_c \tilde{C}_r]$$

Then the closed-loop damping matrix (in modal coordinates) will be

$$D = \begin{bmatrix} \tilde{B}_c A_c M_c S_c \tilde{C}_c & \tilde{B}_c A_c M_c S_c \tilde{C}_r \\ \tilde{B}_r A_c M_c S_c \tilde{C}_c & \tilde{B}_r A_c M_c S_c \tilde{C}_r \end{bmatrix} = \begin{bmatrix} D_{cc} & D_{cr} \\ D_{rc} & D_{rr} \end{bmatrix} \quad (7\text{-}7)$$

The rank of D cannot be greater than n_c, the number of critical modes (without the constraint Eq. (7-6), it cannot be greater than the minimum of n_a and n_s); thus it is only possible to exercise "independent" control over the damping of at most a set of n_c modes. However, when n_c is strictly less than n_a and/or n_s, there are additional free parameters in A_c, S_c which can be used to couple the critical and residual modes in such a way that the stable properties of the critical modes are "inherited" by the residual modes.

A general procedure for choosing A_c and S_c is not yet available; the output-feedback problem offers considerably less flexibility in this regard than in the dynamic compensation case. The objectives of the procedure, in order of importance, appear to be

(1) Make $D_{cc} = M_c$.

(2) Guarantee stability of Eq. (7-3), for any M_c which stabilizes the critical modes, and at least for the value of M_c to be determined later.

(3) Attempt to symmetrize and block-diagonalize D by making the off-diagonal terms (D_{cr}, D_{rc}) as small as possible, subject to the stability requirement (2).

Preliminary work indicates that

(1) If $n_c = n_s = n_a$, it is best to colocate sensors and actuators and to choose $A_c = \tilde{B}_c^{-1}$ and $S_c = \tilde{C}_c^{-1}$.

(2) When $n_s = n_a = n$, the objectives are attainable.

(3) The objectives may be generically attainable for n_s and n_a approximately equal to $2n_c$.

(4) The objectives were attained in the example of Section 7.3.

A problem of this essential character must be solved in any output-feedback approach, if the design model is to be based only on critical mode dynamics. Further research is required on this problem.

The remaining steps of the procedure are illustrated, assuming that A_c and S_c have been determined and that $D_{cc} = M_c$. The critical modes are then governed by (see Eq.(7-3))

$$\ddot{\eta}_c + M_c \dot{\eta}_c + \Omega_c^2 \eta_c = [\Phi^T F_e - \tilde{B}A_c M_c S_c e]_c - D_{cr} \dot{\eta}_r \qquad (7\text{-}8)$$

and the residual modes are governed by

$$\ddot{\eta}_r + D_{rr} \dot{\eta}_r + \Omega_r^2 \eta_r = [\Phi^T F_e - \tilde{B}A_c M_c S_c e]_r - D_{rc} \dot{\eta}_c \qquad (7\text{-}9)$$

The synthetic measurements are

$$\bar{y} = S_c y = S_c \tilde{C}_c \dot{\eta}_c + S_c \tilde{C}_r \dot{\eta}_r + S_c e \qquad (7\text{-}10)$$

Equations (7-8) - (7-10) are the equations from which a truncated model is derived.

7.1.2.3 Estimation of Stochastic Effects

The truncated control problem takes the form*

$$\begin{aligned} \ddot{\eta}_c + \Omega_c^2 \eta_c &= \bar{u} + \xi_c \\ \bar{u} &= -M_c \bar{y} \\ \bar{y} &= \dot{\eta}_c + \nu_c \end{aligned} \qquad (7\text{-}11)$$

where

$$\xi_c \tilde{\equiv} [\Phi^T F_e]_c$$

$$\nu_c \tilde{\equiv} S_c (\tilde{C}_r \dot{\eta}_r + e)$$

are to be approximately represented as "equivalent" white noise processes. For this purpose, an estimate of $\| \dot{\eta}_r \|$ is required; this is based on Eq. (7-9) ignoring critical-mode coupling, i.e.

*We have taken $[\tilde{B}A_c M_c]_c = M_c$ and $S_c \tilde{C}_c = I$ in Eq. (7-8) through (7-10); in Eq. (7-8), D_{cr} is given by Eq. (7-7). In writing Eq. (7-11), no terms have been eliminated from Eq. (7-8).

$$\ddot{\eta}_r + \Omega_r^2 \eta_r = [\Phi^T F_e]_r \tag{7-12}$$

or

$$\ddot{\eta}_r + D_{rr} \dot{\eta}_r + \Omega_r^2 \eta_r = [\Phi^T F_e]_r - [\tilde{B} A_c M_c S_c e]_r \tag{7-13}$$

While Eq. (7-12) is more crude, it does not require prior knowledge of the gains M_c and allows the residual mode velocities to be estimated individually. Appropriate estimation procedures are well-known for the cases when the disturbance F_e is specified as a correlated noise process, or when initial condition statistics on $\eta_r(0)$ are specified. The reader is referred to Section 7.3 for an example of how these procedures may be carried out. It should be observed that in Eq. (7-11), ξ_c and ν_c will in general be correlated equivalent noise processes since they are both derived from F_e (and also since a term involving $\dot{\eta}_c$ has been dropped from Eq. (7-12)); in the following subsection they are taken to be independent white noise processes mainly to simplify the exposition of ideas - in some circumstances, such an approximation can be justified. The truncated model is only of order n_c, the number of critical modes, but its parameters depend on prior estimates of residual mode shapes (through Φ in Eq. (7-7)) and frequencies (through Ω_r^2 in Eq. (7-12)). This is the practical constraint imposed on the design procedure.

7.1.2.4 Determination of Optimum Gains

The truncated control problem may be stated as follows: Given a truncated model in state-space form

$$\begin{aligned} \dot{x} &= Ax + B\bar{u} + v \\ \bar{y} &= Cx + w \end{aligned} \tag{7-14}$$

where v and w are independent white noise processes, with

$E\{v(t)\, v^T(\tau)\} = V\, \delta(t-\tau)$ (assumed positive semidefinite)

$E\{w(t)\, w^T(\tau)\} = W\, \delta(t-\tau)$ (assumed positive definite)

find the output-feedback gains in the control law

$$\bar{u} = M\bar{y} \tag{7-15}$$

so as to minimize the asymptotic mean-square response measure

$$J(M) = \lim_{t\to\infty} E\,\{x^T(t)\; Q\; x(t)\} \tag{7-16}$$

It is assumed that B and C are of full rank, and that (A,B,C) is output-feedback stabilizable in Eq. (7-14).

The necessary conditions for this problem take the form

$$M^* = K_B K_C \tag{7-17}$$

where

$$K_B = (B^T P B)^{-1}\; B^T P$$

$$K_C = X C^T W^{-1}$$

and the matrices P, X are solutions of

$$\pi_B{}^T P \pi_B \pi_C + \pi_C{}^T \pi_B{}^T P \pi_B = Q + PA + A^T P \tag{7-18}$$

and

$$(I-\pi_B) K_C W K_C{}^T (I-\pi_B)^T = -V + K_C W K_C{}^T + AX + XA^T$$

with

$$\pi_B \doteq BK_B \text{ and } \pi_C \doteq K_C C. \tag{7-19}$$

The algorithm proposed for solving Eq. (7-18) - (7-19) is as follows:

(1) Find positive semidefinite symmetric matrices P^0, X^0 and evaluate $\pi_B{}^0$, $\pi_C{}^0$ from Eq. (7-19), so that $(A - \pi_B{}^0 \pi_C{}^0)$ is strictly stable. Set $i = 0$.

(2) Solve the Lyapunov equations

$$P^{i+1} (A - \pi_B{}^i \pi_C{}^i) + (A - \pi_B{}^i \pi_C{}^i)^T P^{i+1} = -Q$$

and

$$(A - \pi_B{}^i \pi_C{}^i) X^{i+1} + X^{i+1} (A - \pi_B{}^i \pi_C{}^i)^T = -V - \pi_B{}^i \pi_C{}^i \tilde{W} \pi_C{}^{iT} \pi_B{}^{iT}$$

with $\tilde{W} = C^T(CC^T)^{-1} W (CC^T)^{-1} C$, for P^{i+1}, X^{i+1}. Evaluate π_B^{i+1}, π_C^{i+1} from Eq. (7-19), set $i = i+1$ and repeat.

Through partitioning, the $P^{(i+1)}$ equation may be reduced to a Lyapunov equation and the $X^{(i+1)}$ equation may be reduced to a Riccati equation.

Any least-squares performance measure for the output-feedback problem will yield necessary conditions which are comparable to Eq. (7-17)-(7-19). These equations are analogous, respectively, to the equations for the state-feedback gains, the control, and the filtering Riccati equation of the steady-state linear-quadratic - Gaussian problem. The present problem is also of the LQG class; it is a stochastic control problem where the class of admissible control laws is parameterized by M. The unusual feature of the present formulation is that it suggests the structure of the solution for the necessary conditions via the projection operators π_B, π_C. An understanding of the structure of solutions to this problem is absolutely essential if one is to avoid the many shallow local minima which characterize output-feedback problems. The idea is to parameterize the solution set $\{X,P\}$ in such a way that the sequence of solutions $\{X^{(i)}, P^{(i)}\}$ will converge to a global minimum from a prescribed initial guess $\{X^{(0)}, P^{(0)}\}$.

Finally, the parameters of Eq. (7-14) through (7-16) are related to those of the truncated problem Eq. (7-5), (7-11). The following associations can be made

$$x = \begin{bmatrix} \eta_c \\ \dot{\eta}_c \end{bmatrix}, A = \begin{bmatrix} 0 & I \\ -\Omega_c^2 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ I \end{bmatrix}, C = [0 \quad I], M = -M_c$$

$$v \doteq \begin{bmatrix} 0 \\ \xi_c/\|\xi_c\| \end{bmatrix}, V \doteq \begin{bmatrix} 0 & 0 \\ 0 & \|\xi_c\|^2 I \end{bmatrix}$$

$$w \doteq \nu_c/\|\nu_c\| \quad , W \doteq \left[\|\nu_c\|^2\right]$$

where $\|\xi_c\|$, $\|\nu_c\|$ are estimates of the magnitudes of ξ_c, ν_c defined in Eq. (7-11); the appropriate means of estimating these variables depends on the specification of the plant disturbances and sensor and activator accuracy (see the examples, Section 7.3).

If we choose the line-of-sight error due to the critical modes as the response variable, then in Eq. (7-4)

$$r_c = H_c \eta_c \quad \text{and} \quad Q = \begin{bmatrix} H_c H_c^T & 0 \\ 0 & 0 \end{bmatrix}$$

in the truncated problem. Alternatively, if the critical mode deflections themselves are taken as responses, choose $H_c = I$ above.

The design procedure may be iterated to yield improved performance. If M_c^* is the first set of gains produced by the algorithm, one can return to Section 7.1.2.2, improve the truncated model (now M_c is known in Eq. (7-7)), perform the estimation of the noise covariances (using Eq. (7-13)), and recompute a new set of gains M_c^{**} using the improved parameters. At present, the convergence of this iterative procedure can only be conjectured; it is plausible that the procedure will converge because the effects of the residual modes were over-estimated (from Eq. (7-12)) in the first iteration, causing the gains to be somewhat larger than necessary—in succeeding stages, the effects of residual modes and the gains should be gradually reduced. This then, completes the outline of the general SOFC method applied to the large space structure control problem.

7.1.3 Summary of Assumptions and Technical Innovations

The key assumptions of the SOFC method have been stated at appropriate places in the previous subsection; only the key assumptions are restated here:

(1) The high-order model of the structure is sufficiently accurate that the effects of unmodelled residual modes can be ignored.

(2) The number of available actuators and sensors (n_a, n_s) is sufficiently large relative to the number of critical modes that the structure can be stabilized using synthetic outputs and inputs. Roughly, this requires $n_a \approx n_s \approx 2n_c$, where n_c is the number of critical modes. Alternatively, one can require that the number of modes controllable by output feedback be of the order of 1/2 min $\{n_a, n_s\}$, if n_a, n_s are prespecified.

(3) Deflection rate measurements (as opposed to measurements of deflection itself) are available. (This assumption is removable.)

(4) A characterization of the external disturbance forces, F_e, on the structure is available, along with the modal transformation matrix, Φ of the high-order model and the sensor errors, e. It is preferable, but not essential, that F_e and e be characterized in terms of their statistics.

(5) The local convergence of an algorithm for computing optimal gains which satisfy Eq. (7-17) through (7-19) can be established.

The key technical invovations of the SOFC method applied to the large space structure problem are the procedures for computing a truncated model and the method of computing optimum gains for the truncated model. The method is such that if stabilizing output-feedback gains exist, then the optimum gains will be bounded and will stabilize the truncated model, and the performance will be robust to modelling and truncation errors and the presence of sensor noise.

7.2 Discussion

In this section, some remarks which may be of use in comparing the stochastic output feedback control method with alternative methods are collected. Discussion is restricted to the procedures described explicitly in Section 7.1; however some fundamental improvements are proposed in Section 7.4.3, and these possibilities should also be taken into account in assessing the potential of this approach.

7.2.1 Strengths

The strengths of the SOFC method are perceived to be:

(1) The method is explicit and complete. The procedure of going from performance specifications to optimal gains is fully-specified. At those points where the judgement of the designer is required, the criteria by which decisions are to be evaluated are clear. The algorithmic requirements of the method are clear.

(2) The method is flexible. It is readily extended to take into account additional design requirements such as:

(a) Incorporation of sensor, actuator and disturbance dynamics.

(b) Different selections of critical modes, or choices of sensor and actuator complements.

(c) Sensor signals which include mode deflections as well as velocities can be incorporated.

(d) Changes in performance criteria.

(e) Generalization to design of low-order compensator dynamics.

(3) The method is robust to modelling errors and stochastic effects, though it can be "tuned" to provide the best performance, e.g., when the key performance objectives are very clear, or when the disturbance models are very accurately known. The method can be expected to yield reasonable values of feedback gains, which are not highly sensitive to the parameters of the truncated model.

(4) The method is "state-of-the-theory." It incorporates some of the most recent advances in thinking about multivariable control problems by theoreticians. It takes into account, in at least an approximate way, issues such as model aggregation, modal spillover, robustness and uncertainty, the algebraic complexity of the output-feedback problem, and the physics of flexible vehicles.

(5) The method provides qualitative insights which are useful to the designer. For instance, it can be seen from Eq. (7-7) that independent control of all critical and residual modes is not generally possible. Coupling of critical and residual modes may actually enhance stability, but it may in general be asymmetric and will tend to make the designers' task more difficult by destroying the identity of the critical modes in the closed-loop system. Approximate decoupling, on the other hand, will require a number of sensors and actuators roughly equal to twice the number of critical modes, and will be enhanced by colocation of sensors and actuators. A number of other insights are provided.

7.?.2 Weaknesses

The weaknesses of the method (aside from immaturity, as discussed in the next subsection), are perceived to be:

(1) Computational requirements. The computational requirements of the algorithm [following Eq. (7-19)] are considered to be reasonable. However, the computational requirements for the model truncation procedure (in particular, finding A_c and S_c so that the closed-loop damping matrix Eq. (7-7) has the desired properties) may be heavy. At worst, this procedure would involve the solution of a numerical optimization problem in n_c $[(n_s-n_c) + (n_a-n_c)]$ parameters involving the solution of a 2n-th order eigenvalue problem at each iteration, where n is the total number of critical and residual modes, n_c is the number of critical modes, n_s is the number of sensors and n_a is the number of actuators. Typical values might be $n = 100$ to 1000, $n_c = 30$, $n_s \simeq n_a \simeq 60$. The best case for this truncation calculation is not yet known.

(2) No prior guarantee of closed-loop stability. Because of the approximations involved in determining a reduced-order model, closed-loop stability of all modes cannot be absolutely guaranteed: the design values must be implemented and tested on a high order model to test stability. The predicted damping imparted to the critical modes of the design model will be fairly accurate, but the behavior of the residual modes is not completely predictable. The optimal gains, whenever they exist, are guaranteed to stabilize the truncated model.

(3) No prior guarantee of existence of optimum gains. The algorithm proposed for determination of optimum gains is not guaranteed to converge; however, it is possible that a convergence proof can be developed.

(4) No prior guarantee of traditional design specifications. The SOFC method does directly minimize mean-square line-of-sight error. However, it is not guaranteed to yield closed-loop pole positions which conform to the preconceived notion of the designer, nor to traditional design specifications such as gain margins, phase margins, or settling times. It is not guaranteed to meet control energy constraints either. Methods are available to incorporate these objectives as "soft" constraints, but ultimately the gains produced must be tested on a system model to see if "hard" requirements are met.

7.2.3 Maturity

The procedure described in Section 7.1 has not been fully implemented, except for the example of the following Section; therefore it must be regarded as immature. It is possible, judging from previous experience on similar problems, that unforeseen pitfalls may be encountered. On the other hand, the method does benefit from the "state-of-the-theory" which is experience of a vicarious sort. On the basis of current information, it can only be said that the method is expected to be computationally feasible, and that the potential benefits appear to be quite substantial.

7.2.4 Applicability to Control of Large Space Structures

The model truncation procedure has been developed specifically for application to large space structure control problems, and it exploits the properties of the physical equations of motion for such structures. The procedure for computing gains has been developed specifically for design of low-order compensators (in the present case, zero-order compensation) for high-dimensional systems. The particular attributes of the physical equations have been carried through the calculations of the next section, but not through the general procedure of Section 7.1; preliminary derivations indicate that substantial simplifications of the necessary conditions Eq. (7-17) through (7-19) can be made in this case. It can be concluded that the method, by design, is suitable for control of large-space structures.

The reader is referred to Section 7.4 for concluding remarks.

7.3 Illustration: Application of the Stochastic Output Feedback Control (SOFC) Method to Control of a Two-Mode System

7.3.1 Introduction

A two-mode system for evaluation of candidate large-space-structure design methods is presented in Section 2.5. The purpose of this subsection is to describe the application of the candidate method to this example. The essential idea of the SOFC method is to determine feedback gains to minimize steady-state mean-square error responses of the closed-loop stochastic system. In order to formulate the example problem in this manner, two preliminary calculations were performed. First, a model truncation procedure was carried out in order that dynamic calculations need be based only on the critical mode and not on the detailed behavior of the unmodelled residual mode. While this procedure normally would not be applied to a low-order example, it must be applied in the large-scale problem because of the need to guarantee stability of residual modes in the controlled system. Secondly, the effects of the specified disturbances and residual modes were represented as white noise in the stochastic model. Finally, the SOFC method was applied to determine a single output feedback gain (critical mode damping) of the aggregated model.

7.3.2 Model Aggregation

An implied constraint on the problem is that the dynamic analysis of the design procedure be based only on the critical modes. However, prior knowledge of the residual mode parameters (e.g., frequencies, eigenfunctions - in this case f_1, ϕ_1) may be assumed. It should be remarked that in general this is a very severe constraint, and it is doubtful whether absolute prior guarantees of stability are possible for any design method, short of actually working out the closed-loop eigenvalues of the full system. However, it is reasonable to seek design methods which "acknowledge" the presence of the residual modes, are robust to their effects, and provide design parameters whereby an initial design may be tuned-up.

The "aggregate model" presented here, as with many alternative schemes, is based on retaining the open-loop critical mode dynamics. However, we have available, then, two inputs (F_1, F_2) and two outputs ($\dot{q}_1$, $\dot{q}_2$) to control one mode. In the present case it can be shown that there is never any advantage to using more than one input and one output to control a single critical mode.* Hence we synthesize a combined actuator signal, u_c, such that

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \bar{u}_c = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \qquad (7\text{-}21)^{**}$$

*This is due to the particular structure of the modal equations - in general, a second order system would require no more than one input and output.

**Eq. (7-20) has been deleted.

and a combined sensor signal, $\bar{y}_c$, such that

$$\bar{y}_c = \begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \tag{7-22}$$

The weighting factors (α_1, α_2, β_1, β_2) are viewed as design parameters. They may be chosen on the basis of prior knowledge of the residual modes. Their choice will affect the extent to which residual mode disturbances must be accounted for in the critical-mode design model, and also the degree to which the critical modes will couple into the residual modes.

Usually the coupling of the residual and critical modes in the closed-loop system is viewed as being undesirable; however in this case it must be exploited: with only one (combined) input and output, it can be shown that we only have independent control, in effect, of one (combined) mode. The idea is to couple the critical and residual modes in such a way that stabilization of the critical mode will also guarantee stabilization of the residual mode. In the present example, this can be accomplished directly using Routh's criterion; in the general case, more sophisticated procedures would be required (e.g., minimizing the spectral norm of the closed-loop system matrix with respect to the weighting parameters). In the present example, it is readily verified that there are some choices of the weighting factors for which no choice of feedback gain is stabilizing (see Section 7.5.2); hence this procedure is quite critical and must be performed with care.

Now the calculations are carried out. Suppose that the feedback law is

$$\bar{u} = -M_c \, \bar{y} \tag{7-23}$$

where M_c is the feedback gain (positive-valued) to be determined later. Then the closed-loop system defined by Eq. (7-2) and (7-21) through (7-23) is given by

$$\ddot{\eta} = \Omega^2\eta - \Phi^T \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} M_c \, [\beta_1 \;\; \beta_2] \, \Phi \, \dot{\eta} = -\Omega^2\eta - D\dot{\eta}$$

The damping matrix, D, is then explicitly

$$D = \begin{bmatrix} (\phi_{11}\alpha_1 + \phi_{21}\alpha_2) M_c (\phi_{11}\beta_1 + \phi_{21}\beta_2) & (\phi_{11}\alpha_1 + \phi_{21}\alpha_2) M_c (\phi_{12}\beta_1 + \phi_{22}\beta_2) \\ (\phi_{12}\alpha_1 + \phi_{22}\alpha_2) M_c (\phi_{11}\beta_i + \phi_{21}\beta_2) & (\phi_{12}\alpha_1 + \phi_{22}\alpha_2) M_c (\phi_{12}\beta_1 + \phi_{22}\beta_2) \end{bmatrix}$$

We want to choose the α's and β's so that: (1) $D_{22} = M_c$ (i.e., the desired critical mode damping), and (2) the closed-loop system is stable. From (1), we find that

$$\alpha_2 = \phi_{22}^{-1} (1 - \phi_{12} \alpha_1)$$
$$\beta_2 = (1 - \beta_1 \phi_{12}) \phi_{22}^{-1} \tag{7-24}$$

From (2) we find that, in view of Eq. (7-24),

$$D_{11} = \left[\phi_{21} \phi_{22}^{-1} + \left(\phi_{11} - \phi_{21} \phi_{12} \phi_{22}^{-1}\right) \alpha_1\right] M_c \left[\phi_{22}^{-1} \phi_{21} + \left(\phi_{11} - \phi_{12} \phi_{21} \phi_{22}^{-1}\right)\beta_1\right]$$
$$= (C_1 + C_2 \alpha_1) M_c (C_1 + C_2 \beta_1)$$

where $C_1 = 1.6603$, $C_2 = 1.9389$ are known. Proceeding further, we find that $D_{12} = C_1 + C_2 \alpha_1$ and $D_{21} = (C_1 + C_2 \beta_1)$. Further examination of the damping matrix, D, reveals that it has rank 1 (as claimed above), but that the residual mode damping with coupling ignored, D_{11}, can be made positive by at least some choices of α_1, β_1 (precisely, for α_1 and β_1 both less or both greater than -0.856). Furthermore, when

$$(C_1 + C_2 \alpha_1) = (C_1 + C_2 \beta_1) = \gamma \tag{7-25}$$

the damping matrix will be symmetric, which implies that the "essential character" of the open-loop modes will be retained in the closed loop system. Working out the closed-loop system matrix under the assumptions of Eq. (7-24) and (7-25), we find

$$A_{c\ell} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega_1^2 & 0 & -M_c \gamma^2 & -M_c \gamma \\ 0 & -\omega_2^2 & -M_c \gamma & -M_c \end{bmatrix}$$

where $\omega_{1,2} = 2\pi f_{1,2}$. The Routh array is

$$\begin{array}{lll}
1 & \left(\omega_1^2 + \omega_2^2\right) & \omega_1^2 \omega_2^2 \\[2ex]
M_c (\gamma^2 + 1) & M_c (\omega_2^2 \gamma^2 + \omega_1^2) & 0 \\[2ex]
\dfrac{\omega_2^2 + \gamma^2 \omega_1^2}{(\gamma^2 + 1)} & \omega_1^2 \omega_2^2 & 0 \\[3ex]
\dfrac{M_c (\gamma^2 + 1) \gamma^2 (\omega_2^4 + (\gamma^2 - 1) \omega_1^2 \omega_2^2 + \omega_1^4)}{\left(\omega_2^2 + \gamma^2 \omega_1^2\right)} & 0 & 0 \\[3ex]
\omega_1^2 \omega_2^2 & 0 & 0
\end{array}$$

We did not optimize γ, but observed that even though D is singular, there are some values of γ which guarantee closed-loop stability for any choice of $M_c > 0$. In particular, this is true for $\gamma = 1$, which is the value we use subsequently.

A very significant observation should be made at this point! The value $\gamma = 0$ completely decouples the residual modes in the closed-loop system, but by the same token (as is apparent from the fourth row of the Routh array!) it does not guarantee strict stability, because (obviously) the residual modes are assumed to have no damping. Thus, although modal control (of mode 2) is our stated objective, it would be entirely incorrect to pick $\gamma = 0$ for the purpose of achieving "perfect" modal control and "perfect" decoupling! The resulting controller would be very highly sensitive to any disturbances of the residual modes (even assuming that Φ and Ω^2 were perfectly known - which they are not), and would exhibit very undesirable responses (e.g., to $F_2(t) = \sin 3t$). This is a very easy trap to fall into.

The more appropriate value, $\gamma = 1$, guarantees closed-loop stability and yields

$$\begin{aligned} \alpha_1 &= \beta_1 = -0.3407 \\ \alpha_2 &= \beta_2 = 1.9398 \end{aligned} \tag{7-26}$$

with

$$D = M_c \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

To summarize, the problem is now to design M_c for the system

$$\begin{aligned} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} &= \begin{bmatrix} -\omega_1^2 & 0 \\ 0 & -\omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bar{u} + \begin{bmatrix} 0.606 \\ 0.365 \end{bmatrix} F_2(t) \\ \bar{y} &= [1 \quad 1] \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} \end{aligned} \tag{7-27}$$

7.3.3 Modelling of Noises and Disturbances

Starting from Eq. (7-27), we can extract the critical mode equation

$$\begin{aligned} \ddot{\eta}_2 &= -\omega_2^2 \eta_2 - M_c \bar{y} + 0.365 F_2(t) \\ \bar{y} &= \dot{\eta}_2 + \dot{\eta}_1 \end{aligned} \tag{7-28}$$

The two disturbances are

(1) An initial condition $q_2(0) = 1$, which implies

$$\eta(0) = \begin{bmatrix} 1.21 \\ 0.729 \end{bmatrix}, \quad \dot{\eta}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7-29}$$

and (2) A steady disturbance

$$F_2(t) = \sin 3t$$

In Eq. (7-28), we are not permitted to make an explicit model of the output disturbance $\dot{\eta}_1$.

Our very crude approach is to approximate $F_2(t)$ and $\dot{\eta}_1(t)$ in Eq. (7-28) by white noise processes. The intensity of these processes is based on the disturbances (1), (2), and the behavior of the uncontrolled system. Thus, the design model will take the form

$$\begin{aligned} \dot{x} &= Ax + Bu + v \\ y &= Cx + w \end{aligned} \tag{7-30}$$

where

$$x = \begin{bmatrix} \eta_2 \\ \dot{\eta}_2 \end{bmatrix}, \quad u \approx \bar{u} \quad , \quad y \approx \bar{y}$$

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_2^2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [0 \quad 1]$$

and v, w are independent white noise processes with (formally)

$$E\left\{v(t)\right\} \equiv 0 \quad , \quad E\left\{v(t)\, v^T(\tau)\right\} = V\delta(t - \tau)$$

$$E\left\{w(t)\right\} \equiv 0 \quad , \quad E\left\{w(t)\, w^T(\tau)\right\} = W\delta(t - \tau)$$

In choosing V, W, we would normally look at whether disturbance (1) or (2) were worse and use either a worst-case or average value based on the disturbance magnitudes. In the present case, (2) was judged to be more significant than (1), though precise estimates of the relative importance of the effects were not computed. In the first equation of (7-28), we estimate $|F_2(t)| = 1$ and thus take

$$V = \begin{bmatrix} 0 & 0 \\ 0 & V_{22} \end{bmatrix}$$

with

$$V_{22} = (0.365)^2 \; |F_2(t)| = (0.365)^2$$

and in the second equation of (7-28) the steady-state driven response to

$$\ddot{\eta}_1 = - \omega_1^2 \eta_1 + (0.606) \sin 3t$$

takes the form

$$\eta_1(t) = A \sin 3t$$

with

$$A = (0.606)/(\omega_1^2 - 9)$$

Thus

$$\dot{\eta}_1 \sim 3A \cos 3t$$

and $|\dot{\eta}_1| \sim |3\,(0.606) \div (\omega_1^2 - 9)| = 0.2089$. Thus we took $W = 0.0436 = (0.2089)^2$. These approximations are so crude that evidently different estimates could be made. In the results of the next section, V_{22} and W are carried as parameters so that the effects of different choices can be evaluated. Note that we have already guaranteed stability for any (positive) value of M_c.

7.3.4 Application of the Stochastic Output Feedback Approach

The stochastic formulation of the output feedback problem captures an essential idea of classical control design: if the gain M_c is chosen too small, the effects of the plant disturbances, v, will be significant; if M_c is too large, the output disturbances, w, will create problems - thus the best value of M_c will represent a compromise between these two extremes.

More specifically, we seek M_c such that

$$J(M_c) = \lim_{t \to \infty} E \left\{ x^T(t)\, Q\, x(t) \right\} \tag{7-31}$$

is minimized (for some positive definite symmetric matrix Q) subject to the dynamic equation (7-30). Notice that if $Q = H_c^T H_c$, then Eq. (7-31) represents the sum of the mean-square values of the responses $r = H_c x$. We chose Q in the form

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}$$

The necessary conditions for this problem are (cf. Eq. (7-17) through (7-19))

$$M_C = K_B K_C$$

$$K_B = (B^T PB)^{-1} B^T P \tag{7-32}$$

$$K_C = XC^T W^{-1}$$

where the 2×2 symmetric positive semidefinite matrices P and X are solutions of

$$\pi_B{}^T P \pi_B \pi_C + \pi_C{}^T \pi_B{}^T P \pi_B = Q + PA + A^T P \tag{7-33}$$

and

$$(I-\pi_B) K_C W K_C{}^T (I-\pi_B)^T = -V + K_C W K_C{}^T + AX + XA^T$$

and

$$\pi_B \doteq BK_B \text{ and } \pi_C \doteq K_C C. \tag{7-34}$$

Letting

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}, \quad X = \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix}$$

it is possible to show algebraically that

$$M_C{}^* = \frac{1}{W}\left[\frac{p_{12}\, x_{12}}{p_{22}} + x_{22}\right]$$

where

$$x_{12} = 0$$

$$x_{22} = \sqrt{V_{22}\, W} \quad \text{(independent of q)}$$

and thus

$$M_c{}^* = \sqrt{V_{22}/W} = 1.7480$$

Evidently, this incorporates the essential aspects of the tradeoff mentioned earlier in this section.

The predicted damping ratio for the critical mode is then

$$\zeta = \frac{M_c^*}{2\omega_2} = \frac{1.748}{4\pi(0.412)} = 0.338$$

Finally, we also used the same methods to work out another test case where $\alpha_1 = \beta_1 = 0$, $\alpha_2 = \beta_2 = 1$ are not optimally determined but are stabilizable and correspond to a "common sense" approach to the choice of weightings for the given disturbances, i.e., we control and measure q_2 because the disturbances act on it. These results are given in Section 7.5.1.

7.3.5 Conclusions

First we answer the specific questions posed in Section 2.5:

(1) The computation of the gain matrix is described in Section 7.3.4. The SOFC method itself involves the solution of Eq. (7-32) through (7-34), which can be carried out analytically for the present example.

(2) The method guarantees stability of the residual modes via Routh's criterion. For the choice of parameters used here, the residual and critical mode damping ratios are predicted to be about 1.6 and 0.338 respectively. The actual transient responses are shown on the accompanying illustrations.

(3) Control and observation spillover are <u>essential</u> to achieving stability when combined sensor and actuator signals are employed, as in Section 7.3.2. These effects are accounted for both in the truncation and stochastic modelling.

(4),(5) These questions are answered by the accompanying plots (Figures 7-1, 7-2). The response to initial conditions, (4), was not explicitly designed for, although a more refined estimation of the stochastic terms might have improved this response somewhat.

(6) There are limitations on the damping of all modes which are imposed primarily by the use of combined sensors and actuators rather than the SOFC method itself. By appropriate choice of noise statistics V_{22}, W, any desired damping can be achieved for the design model involving only the critical mode. However, this will not be the actual damping in the full closed-loop system. The "identity" of the modes is preserved under the type of feedback proposed here, whereas it may not be with other types of feedback.

(7) A variation of the method can be applied (see Section 7.5.1) so long as the number of velocity sensors and force actuators is <u>not less</u> than the number of critical modes.

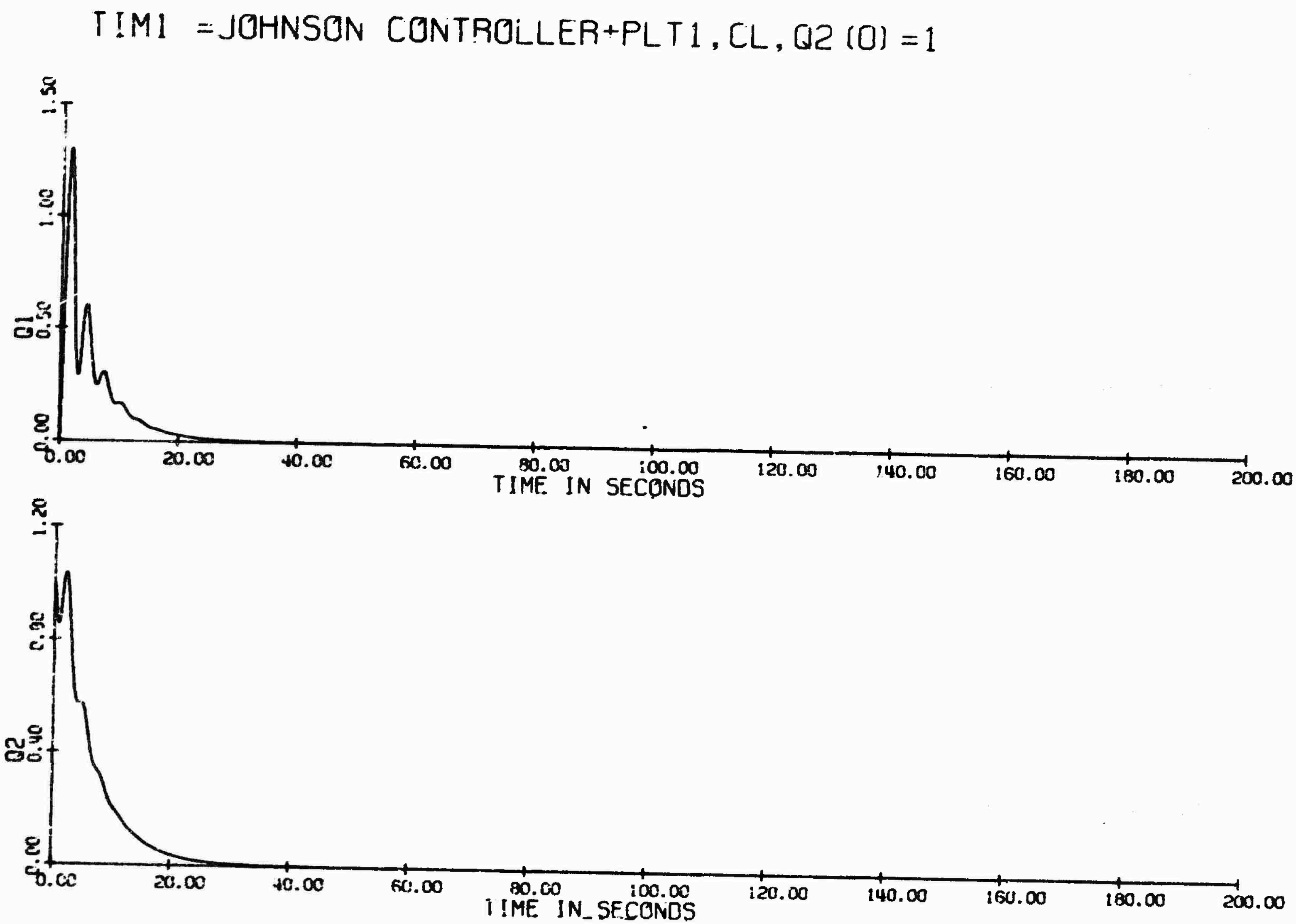


Figure 7-1. Response to initial conditions $q_2(0) = 1$.

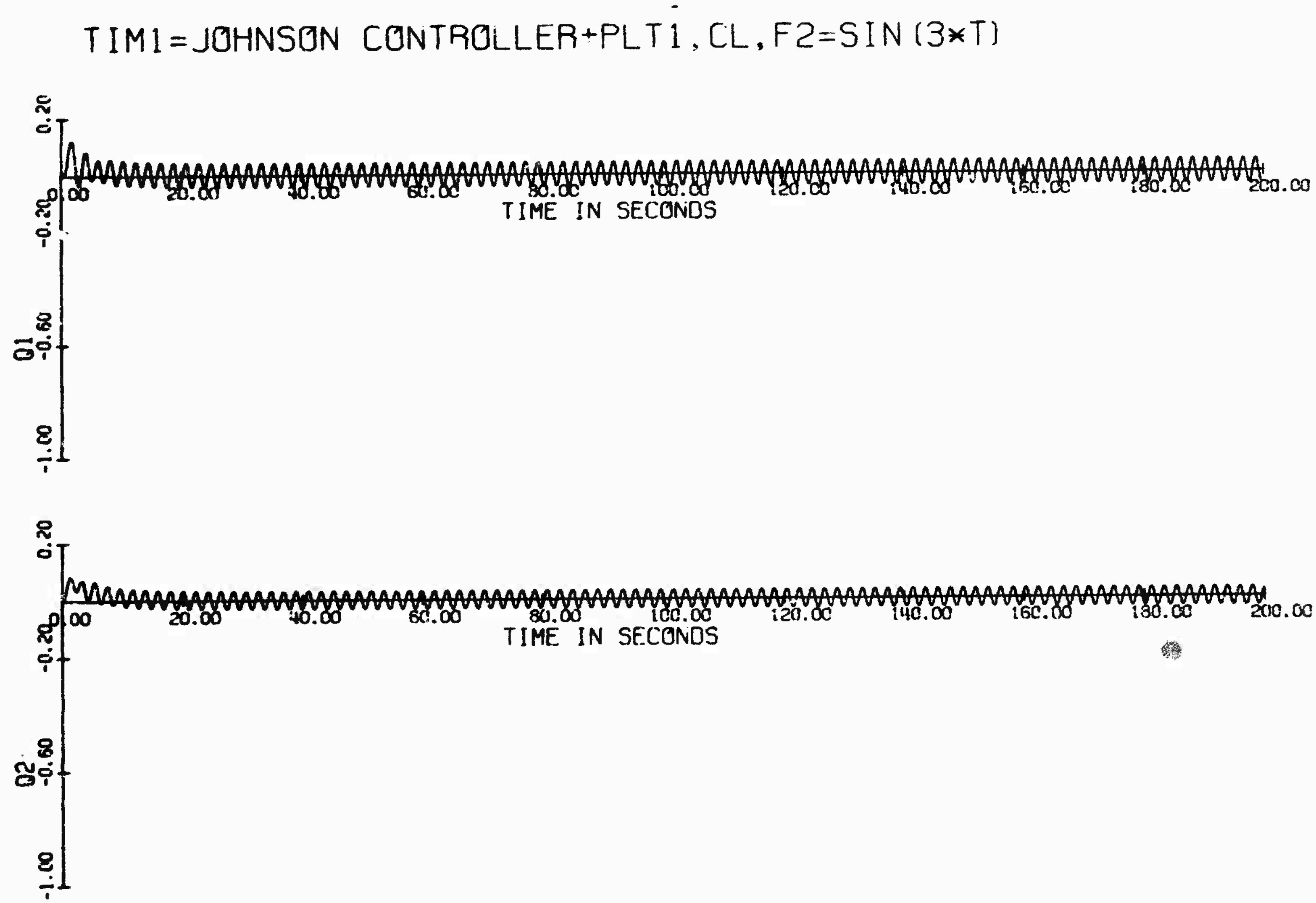


Figure 7-2. Response to periodic disturbance $f_2(t) = \sin 3t$.

(8) In principle, the proposed approach can be carried through for any combination of position and velocity sensors.

We conclude with a few remarks about the possible relative merits of the proposed approach. The approach involves several approximations and probably can be improved, even in the context of the present example; under the imposed design constraints it is doubtful whether there exists any procedure which is truly "optimal" in the absence of complete knowledge of all of the residual modes. However, the methodology does generalize to the case of a large number of critical and residual modes and would probably yield results qualitatively similar to those obtained in the present case.

The aspects of the example which require further investigation are:

(1) The algorithm for computing optimum sensor and actuator combinations.

(2) The modelling of disturbances due to residual modes in the design model.

(3) Proposed computational algorithms to solve the SOFC equations for higher order systems need to be coded and tested.

None of these tasks is currently expected to be insurmountable, though undoubtedly additional issues will arise.

7.4 Conclusions

7.4.1 Summary of Advantages (see Section 7.2)

(1) Explicit and complete.

(2) Flexible.

(3) Robust.

(4) Incorporates recent theoretical advances.

(5) Qualitative insights.

(6) Well-suited to large-space-structure control problem.

7.4.2 Summary of Disadvantages (see Section 7.2)

(1) Computational requirements might be heavy.

(2) No prior guarantee of closed-loop stability.

(3) No prior guarantee of existence of optimum gains.

(4) No prior guarantee of meeting traditional design specifications.

(5) Immature.

7.4.3 Final Comments

Naturally, one must ask whether the disadvantages are insurmountable, and if not, whether they outweigh the advantages. We are inclined to answer these questions both in the negative. The disadvantage 7.4.2(1) of heavy computational requirements is likely to be surmountable, although no radical innovations are apparent. A likely possibility is to apply available techniques for estimating the eigenvalue of a large matrix having the largest real part. The estimation can be used with the guidelines of Section 7.1.2.2 to yield an accuarate initial guess for the sensor and actuator combinations A_c, S_c, so that the large eigenvalue problem

$$\det \begin{bmatrix} \lambda I & - Iu \\ + \Omega^2 & \lambda I + D \end{bmatrix} = 0$$

need only be solved once or twice. The disadvantages 7.4.2 (2), (4) are felt to be inherent to the SOFC method described in Section 7.1, but they are not fatal; the designed gains must simply be tested on the full model to determine transient response and disturbance rejection properties. A procedure for iterative improvement of the initial gain estimates has been described. Disadvantage 7.4.2(3) may or may not be inherent to the method; however, a feasible numerical algorithm is almost surely guaranteed in some cases; the design example yielded reasonable results. Note that the truncated problem is generically stabilizable. The only solution to the problem of immaturity is to gain some practical experience.

The potential advantages, by contrast, are substantial. These have been adequately described in Sections 7.1 and 7.2. Many of the advantages, moreover, cover the pitfalls of alternative methods. Thus, although there is a risk of misallocating resources by pursuing the SOFC method, there is also a possibility of covering unforeseen disadvantages of alternative methods.

As a final remark, it must be noted that there are several potential ways of changing and improving the SOFC method presented in Section 7.1 which would merit further investigation. The two most basic areas for improvement are the possibility of a rigorous approach to model aggregation (to replace the ad hoc truncation procedure), the incorporation of control-energy penalties in the performance index, and the extensions to include more general sensor and actuator models. These are all considered to be feasible. The biggest potential payoff lies in the area of model aggregation - this holds forth the possibility of prior determination of closed-loop stabilizability, and requires a fundamentally new development based on asymptotic properties of finite-element approximations as the model order approaches infinity. The other areas of potential improvement merely require extensions of technical details.

7.5 Appendices

7.5.1 SOFC Method Applied to the Case $\alpha_1 = \beta_1 = 0$, $\alpha_2 = \beta_2 = 1$

The approach to this case follows precisely that used in Sections 7.3.4 and 7.3.5. We find the equations

$$\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} = \begin{bmatrix} -\omega_1^2 & 0 \\ 0 & -\omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} + \begin{bmatrix} 0.606 \\ 0.365 \end{bmatrix} \bar{u} + \begin{bmatrix} 0.606 \\ 0.365 \end{bmatrix} F_2(t)$$

$$\bar{y} = [0.606 \quad 0.365] \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix}$$

Proceeding as in Section 7.3.4, we find a design model

$$\dot{x} = Ax + Bu + v$$

$$y = Cx + w$$

with

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_2^2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.365 \end{bmatrix}, \quad C = [0, 0.365]$$

$$X = \begin{bmatrix} \eta_2 \\ \dot{\eta}_2 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}$$

and

$$E\{v(t)\} \equiv 0, \quad E\{v(t)\, v^T(\tau)\} = \begin{bmatrix} 0 & 0 \\ 0 & (0.365)^2 \end{bmatrix} \delta(t-\tau)$$

$$E\{w(t)\} \equiv 0, \quad E\{w(t)w^T(\tau)\} = (0.606)^2(0.0436)\ \delta(t-\tau) = 0.016\ \delta(t-\tau) = W\ \delta(t-\tau)$$

The expressions for calculating M_c are exactly the same as in Eq. (7-32) through (7-34). Working through the algebra, it again turns out that

$$M_c^* = \frac{1}{W}\left[\frac{p_{12}x_{12}}{p_{22}} + x_{22}\right]$$

$$x_{12} = 0$$

$$x_{22} = \frac{\sqrt{V_{22}\,W}}{0.365}$$

so that

$$M_c^* = (0.365)^{-1}\ \sqrt{V_{22}/W} = -2.8845$$

Responses to the two disturbances are shown in Figures 7-3 and 7-4. It is seen that the transient response is quite fast, but that the response to the sinusoidal disturbance, while less than the open-loop response, is relatively large. The closed-loop system is stable.

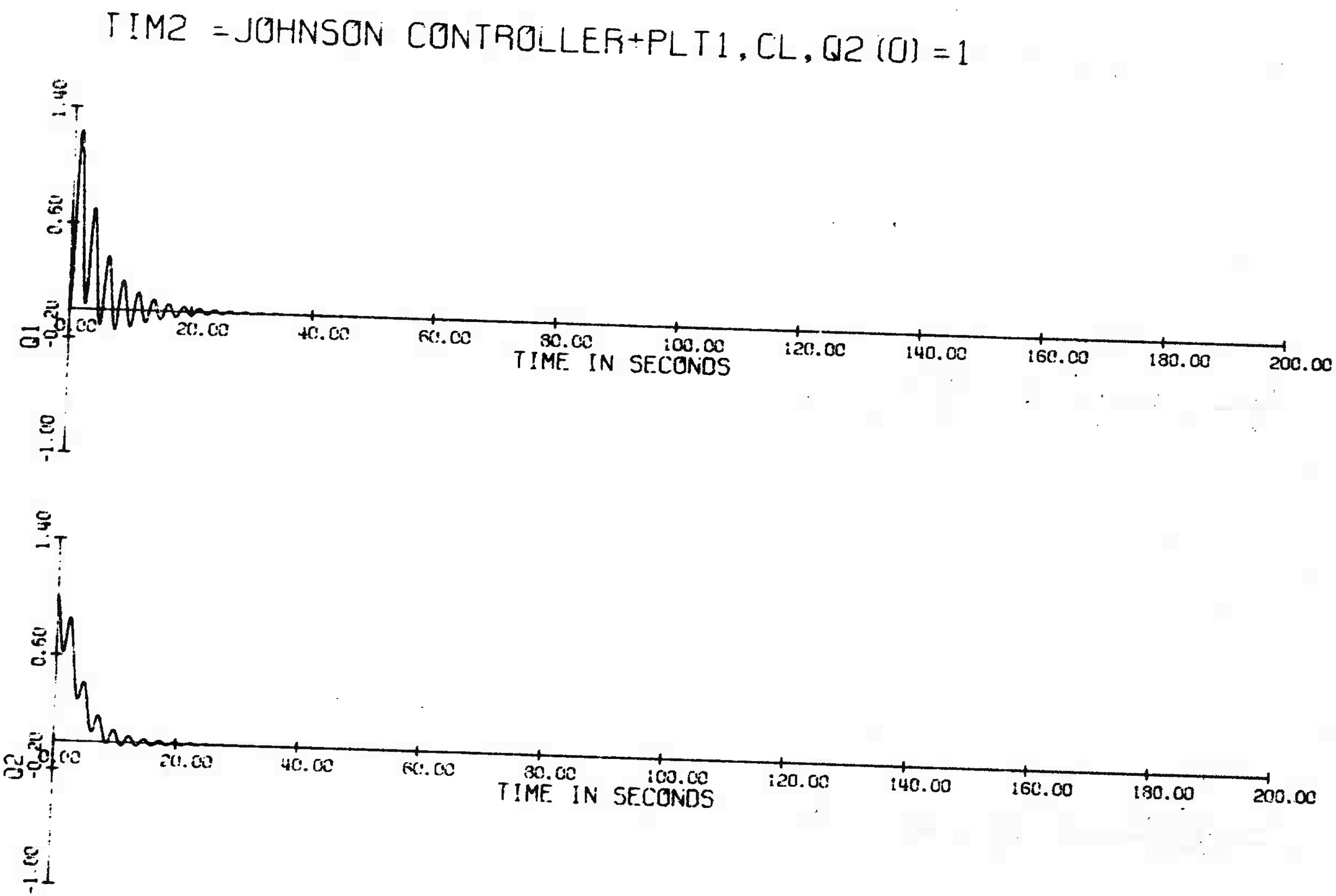


Figure 7-3. Response to initial conditions $q_2(0) = 1$.

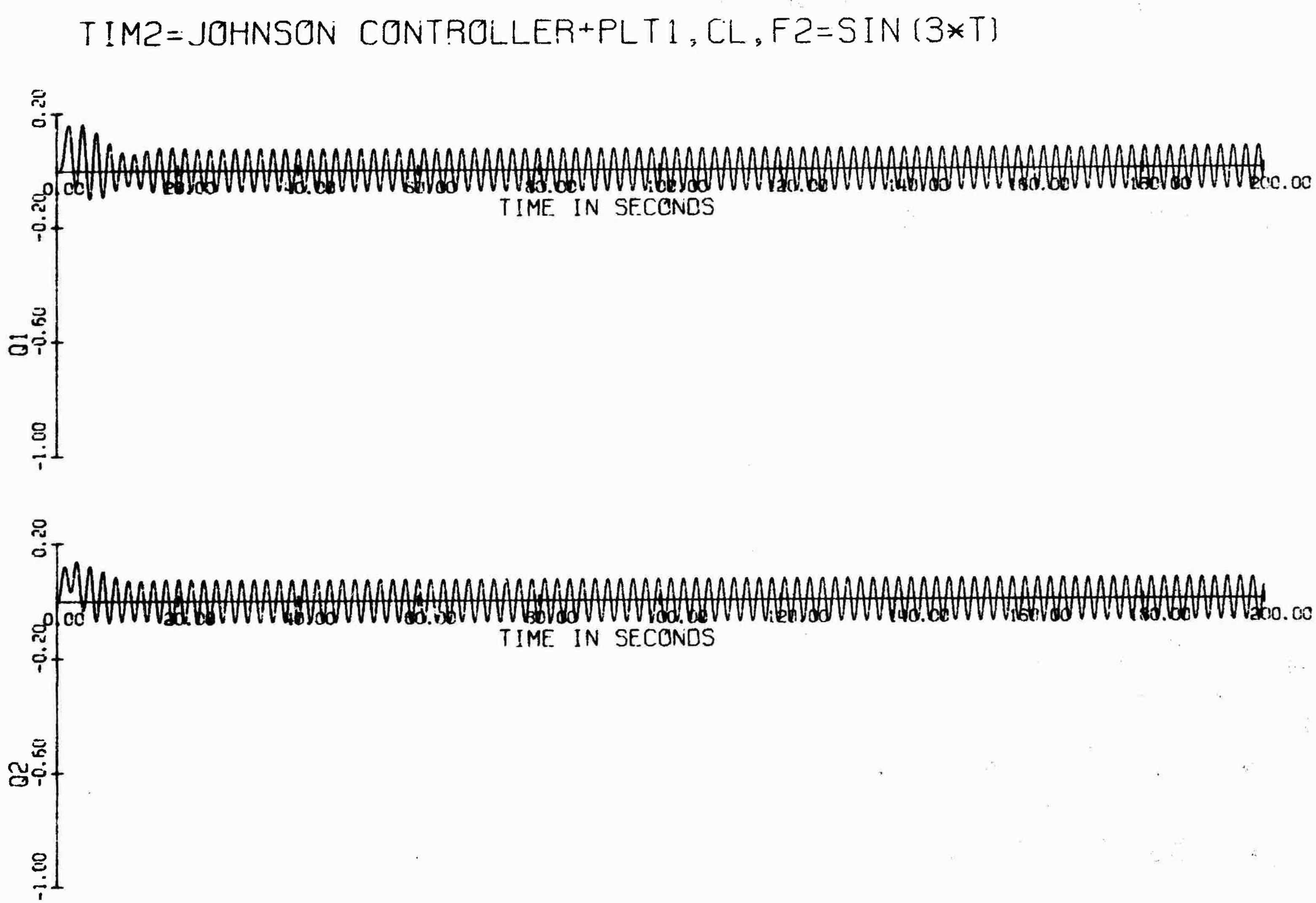


Figure 7-4. Response to periodic disturbance $f_2(t) = \sin 3t$.

7.5.2 A Sensor/Actuator Combination Which Destroys Output-Feedback Stabilizability of the Example

In the course of solving the two-mode example, we came across some values of $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ which destroyed stabilizability by introducing a destabilizing coupling of critical and residual modes. Consider

$$(\alpha_1, \alpha_2, \beta_1, \beta_2) = (-0.584, 1.37, -2.033, -2.033)$$

Corresponding to the differential equation following Eq. (7-23) in Section 7.3.2, we find

$$\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -(0.546)^2 & 0 \\ 0 & -(2.59)^2 \end{bmatrix}}_{-\Omega^2} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0.454\ M_c & -0.199\ M_c \\ 2.28\ M_c & -1.0\ M_c \end{bmatrix}}_{-D} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix}$$

The characteristic polynomial is

$$s^4 + 0.5463\ M_c\ s^3 + 7.006\ s^2 - 2.745\ M_c\ s + 2.0$$

The Routh array is

$$\begin{array}{cccc} 1 & 7.006 & 2 & 0 \\ 0.5463\ M_c & -2.745\ M_c & 0 & 0 \\ 12.03 & 2 & 0 & 0 \\ -2.83\ M_c & 0 & 0 & 0 \\ 2.0 & 0 & 0 & 0 \end{array}$$

Consequently, no value of M_c can achieve stability (i.e., whatever value M_c assumes, there will be sign changes in the first column of the array). In this case the stabilizability hypothesis is violated and no output feedback solution exists.

References

1. Johnson, T.L., "Minimum-Variance Fixed-Form Compensation of Linear Systems," Proc. 18th IEEE Conference on Decision and Control, San Diego, California, January 1979.

2. Newton, G.C., L.A. Gould, and J.F. Kaiser, Analytical Design of Feedback Controls, New York, J. Wiley & Sons, 1961.

3. Axsäter, S., "Suboptimal Time-Variable Feedback Control of Linear Dynamic Systems with Random Inputs," Int. J. Control, Vol. 4, No. 6, pp. 549-566, 1966.

4. Levine, W.S., T.L. Johnson, and M. Athans, "Optimal Limited-State-Variable Feedback Controllers for Linear Systems," IEEE Trans. Automatic Control, Vol. AC-17, No. 6, pp. 785-793, December 1971.

5. Platzman, L.K. and T.L. Johnson, "A LQG Control Problem with Innovations-Feedthrough Solution," IEEE Trans. Automatic Control, Vol. AC-21, No. 5, pp. 721-725, October 1976.

6. Blanvillain, P.J. and T.L. Johnson, "Invariants of Optimal Minimal-Order Observer-Based Compensators," IEEE Trans. Automatic Control, Vol. AC-23, No. 3, pp. 473-474, June 1978.

7. Blanvillain, P.J. and T.L. Johnson, "Specific-Optimal Control With a Dual Minimal-Order Observer-Based Compensator," Int. J. Control, Vol. 28, No. 2, pp. 277-294, 1978.

8. Naeije, W.J. and O.H. Bosgra, "The Design of Compensators for Linear Multivariable Systems," Proc. 4th IFAC Symposium on Multivariable Technological Systems, Fredrickton, NB, Canada, July 1977.

SECTION 8

CONCLUSIONS

8.1 Purpose

Preceding sections (3 through 7) of this Volume contain in-depth studies of the five output-feedback methods selected for evaluation (Section 2) as candidates for controller design tools with large space structures. Each study includes a discussion of strong and weak points, one or more designs with a specific (common) example, and recommendations for further work. We deliberately eschew a pedantic restatement in this section of advantages, disadvantages, and "conclusions" already given for the individual methods. The purpose of this section is to identify some specific directions in which it seems advisable to concentrate future research efforts. In order to do this, we focus on a comparison of the performance of the individual designs against the common test example, and interpret the findings in the light of our general insight into each method obtained by the studies in Sections 3 through 7. This approach exposes certain features of the individual design methods which tend to be overlooked when the methods are studied in isolation. Although evaluation against a single test example cannot render a definitive scientific judgment on the relative merits of these design methods, it does assist in making a rational decision as to where near-term research efforts should be concentrated.

8.2 Performance Comparisons

8.2.1 Data Base

Seven specific stable controller designs for the test example have been reported in previous sections, including two each for the methods of Kosut (Section 6) and Johnson (Section 7). The prinicpal parameters of the closed-loop system for the test example incorporating these designs are summarized in Table 8-1; corresponding characteristics of the open-loop system are also shown. Other reported controller designs which produced an unstable closed-loop system are not discussed further.

Some important features of the closed-loop system response to initial conditions for these designs, together with associated data for the open-loop system, are summarized in Table 8-2. The particular initial condition associated with this table is a positive unit displacement of the outer mass of the two-mass system, with all other system states held at zero. (Time response plots in the physical coordinates have been shown in earlier sections.) Responses to initial conditions on the modal coordinates were also studied for each design, mainly to investigate spillover effects. Such initial conditions, although physically realizable, are, in a physical sense, somewhat contrived. Hence, similar tabular summaries for such initial conditions are not shown, although the results were qualitatively used in formulating conclusions. (Typical responses to modal initial conditions for the two stable Kosut designs are shown in Figures 6-10, 6-11, and 6-4, 6-5, respectively.)

Table 8-1. Principal closed-loop system parameters.

DESIGN METHOD	GAIN MATRIX F ($u = Fy$)	MODE 1 CLOSED LOOP ζ	MODE 1 CLOSED LOOP ω (rad/s)	MODE 2 CLOSED LOOP ζ	MODE 2 CLOSED LOOP ω (rad/s)	REPORT REFERENCE SECTION NUMBER
OPEN-LOOP CHARACTERISTICS	—	0	0.5463	0	2.588	-
MODAL DECOUPLING (Canavin)	$\begin{bmatrix} -0.5051 & 0.2151 \\ 0.2151 & -0.09162 \end{bmatrix}$	0.03062	0.5470	0.1000	2.585	Section 3
POLE ASSIGNMENT (Davison-Wang)	$\begin{bmatrix} 0.0 & 0.0 \\ 0.0 & -38.917 \end{bmatrix}$	15.3485 (REAL ROOTS)	0.6312	0.01822	2.240	Section 4
OPTIMAL OUTPUT FEEDBACK (Levine-Athans)	$\begin{bmatrix} -0.597 & 0.0 \\ 0.254 & 0.0 \end{bmatrix}$	0.07270	0.5481	0.1002	2.580	Section 5
SUBOPTIMAL OUTPUT FEEDBACK (Kosut) (1) One Sensor	$\begin{bmatrix} -0.5971 & 0.0 \\ 0.2540 & 0.0 \end{bmatrix}$	0.07272	0.5481	0.1002	2.580	Section 6
(2) Two Sensors	$\begin{bmatrix} -0.6437 & -0.1096 \\ -0.2961 & -1.293 \end{bmatrix}$	0.7071	0.5463	0.09999	2.589	
STOCHASTIC OPTIMAL OUTPUT FEEDBACK (Johnson) (1) Optimized weighting	$\begin{bmatrix} -0.2029 & 1.155 \\ 1.155 & -6.577 \end{bmatrix}$	1.999 (REAL ROOTS)	0.6741	0.1895	2.097	Section 7
(2) Common sense weighting	$\begin{bmatrix} 0.0 & 0.0 \\ 0.0 & -2.8845 \end{bmatrix}$	0.9995	0.5623	0.06326	2.515	

Table 8-2. Response to initial conditions: $q_2(0) = 1$.

DESIGN METHOD	OVERALL PEAK AMPLITUDE				FIRST OVERSHOOT PEAK				5% SETTLING TIME† (s)				ASYMPTOTIC BEHAVIOR			
	q_1 (m)	q_2 (m)	η_1	η_2	q_1 (m)	q_2 (m)	η_1	η_2	q_1	q_2	η_1	η_2	q_1	q_2	η_1	η_2
OPEN-LOOP CHARACTERISTICS	1.242	1.0	1.212	0.729	-1.243	-0.9909	-1.201	-0.7272	UNDAMPED				UNDAMPED			
MODAL DECOUPLING (Canavin)	0.887	1.000	1.212	0.729	-0.7242	-0.700	-1.077	-0.538	150	156	190	11	→0	→0	→0	→0
POLE ASSIGNMENT (Davison-Wang)	1.537	1.000	1.970	0.729	-	-	-	-0.600	137	146	170	69	→0	→0	→0	→0
OPTIMAL OUTPUT FEEDBACK (Levine-Athans)	0.8964	1.000	1.212	0.729	-0.6528	-0.6171	-0.954	-0.554	64	65	80	11	→0	→0	→0	→0
SUBOPTIMAL OUTPUT FEEDBACK (Kosut) (1) One Sensor	0.8964	1.000	1.212	0.7290	-0.6528	-0.6171	-0.947	-0.557	64	65	76	11	→0	→0	→0	→0
(2) Two Sensors	0.9060	1.000	1.212	0.7290	-0.13	-0.0594	-0.05027	-0.5395	11	7	8	8	→0	→0	→0	→0
STOCHASTIC OPTIMAL OUTPUT FEEDBACK (Johnson) (1) Optimized weighting	1.2965	1.039	1.862	0.729	-	-	-	-0.415	17	19	21	10	→0	→0	→0	→0
(2) Common sense weighting	1.238	1.0	1.423	0.729	-0.143	-0.026	-0.013	-0.615	16	11	8	15	→0	→0	→0	→0

† For purposes of comparison, settling time here refers to the time required for the variable indicated to be damped to an absolute value not exceeding 0.05 $q_2(0)$.

Finally, key features of the frequency response of the closed-loop system for each of the designs are summarized in Table 8-3. Data shown for the specific frequency $\omega = 3$ radians/second allow these tabular data to be related to the time response plots obtained for the disturbance input sin 3t that were shown in earlier sections.

8.2.2 Test Design Results

The modal decoupling method (Canavin) permits free selections of the damping ratio for each critical mode, but has no influence upon the damping ratio for residual modes. This fact dominates the test design, which exhibits the desired damping ratio for the critical mode, but has the lowest value for residual mode damping of all the test designs. As a result, the settling time of the physical coordinates in response to initial conditions exceeds that for any of the other test designs. Moreover, the response to a periodic disturbance exhibits the highest steady-state gain in the vicinity of the open-loop residual mode frequency, and the smallest phase margin, of any of the test designs. The discussion in Section 3 emphasizes the guarantee of stability with this method. In contrast, the relatively poor performance exhibited by the test design largely overshadows the fact that the system is stable over a large range of possible parameter variations, since none of these variations has any significant influence over the residual mode damping ratio.

The pole assignment method (Davison-Wang) does not in general permit placement of all system poles as desired, even for the design model, let alone poles corresponding to residual modes. The implications of these facts upon performance is clearly illustrated by the relatively poor performance of the test design in the time domain. The modal characteristics contrast sharply with those of the Canavin design: not even 1/5 of the desired damping for the critical mode is achieved, whereas the residual mode is extremely over-damped. These two properties combine to produce excessive settling times for the physical coordinates in response to initial conditions—nearly as long as for the Canavin design, and more than twice that of any of the other designs. In addition, the excessively high gains used result in much higher peak amplitudes than for the other designs. The frequency response, however, is quite good, exhibiting no resonance regions and the largest phase margin of all of the designs. Overall, however, the design must be judged unsatisfactory, since the primary design objective ($\zeta_{2c} \geq 0.1$) is not attained.†

Performance of the optimal output feedback (Levine-Athans) design is somewhat improved over, but qualitatively similar to, that of the Canavin design. The principal difference is that in the Levine-Athans design, the residual mode damping is substantially larger, although still small in absolute terms. The lack of performance relative to designs yet to be discussed is attributable largely to the inability of the method to influence the damping of residual modes.

† In fairness, it should be noted that the low order of the design model tends to place the results by this method in an unfavorable light. In a higher order design model, a higher percentage of the poles can be assigned (e.g., 3 of 4; rather than 1 of 2, as in this case), possibly leading to better performance than exhibited here.

Table 8-3. Response to periodic input: $\sin \omega t$, $\omega > 0$ (rad/s).

DESIGN METHOD	STEADY-STATE GAIN (dB)					STEADY-STATE PHASE (deg)					GAIN MARGIN (dB)	PHASE MARGIN (deg)
	$\omega \to 0^+$	$\omega = \omega_1$ †	$\omega = \omega_2$ †	$\omega = 3$	$\omega \to +\infty$	$\omega \to 0^+$	$\omega = \omega_1$	$\omega = \omega_2$	$\omega = 3$	$\omega \to +\infty$		
OPEN-LOOP CHARACTERISTICS	1.94	59.05	31.13	-19.43	-40 dB/dec	0.0	0.0	0.0	+180	+180	-	-
MODAL DECOUPLING (Canavin)	1.94	26.07	-19.00	-21.31	-40 dB/dec	0.0	-87.28	-117.24	-166.99	-180	$+\infty$	5.73
POLE ASSIGNMENT (Davison-Wang)	1.02	-27.00	-39.80	-41.44	-40 dB/dec	0.0	-90.23	-87.89	-95.04	-180	$+\infty$	143.07
OPTIMAL OUTPUT FEEDBACK (Levine-Athans)	1.94	18.58	-17.38	-20.875	-40 dB/dec	0.0	-84.65	-137.56	-166.75	-180	$+\infty$	12.62
SUBOPTIMAL OUTPUT FEEDBACK (Kosut) (1) One sensor	1.94	18.58	-17.38	-21.16	-40 dB/dec	0.0	-84.36	-137.48	-169.7	-180	$+\infty$	12.62
(2) Two sensors	1.938	-1.5	-17.46	-21.24	-40 dB/dec	0.0	-88.0	-123.3	-157.6	-180	$+\infty$	104.1
STOCHASTIC OPTIMAL OUTPUT FEEDBACK (Johnson) (1) Optimized weighting	1.924	-8.72	-30.93	-29.61	-40 dB/dec	0.0	-90.44	-105.80	-118.27	-180	$+\infty$	139.06
(2) Common sense weighting	1.935	-4.41	-18.16	-22.63	-40 dB/dec	0.0	-93.08	-63.59	-139.93	-180	$+\infty$	126.43

† ω_1 and ω_2 refer to the frequencies (rad/s) of modes 1 and 2, respectively, for the open-loop system.

The similarity of the suboptimal output feedback (Kosut) method of minimum error excitation, as published, to the Levine-Athans method is reflected in the fact that the test design by the Kosut method for the stable single sensor configuration is virtually identical to the Levine-Athans design (which used the same sensor configuration). Marked performance improvement is exhibited by the design, applied to the two sensor configuration, which is made possible by the extensiions to the Kosut method developed in this volume. The additional sensor gives rise to free design parameters which are used to eliminate residual mode excitation of the critical mode dynamics, and to set at will the value of the residual mode damping ratio; for demonstration purposes, "optimal" damping for the residual mode is chosen. Settling time of the physical (as well as the modal) coordinates in response to initial conditions is the shortest of all the test designs. In response to a periodic disturbance, the steady-state gain resonance peak at the residual mode frequency is completely eliminated, and the phase margin is increased by an order of magnitude, relative to the single sensor design.

The two designs using the stochastic optimal output feedback (Johnson) method are distinguished by the choice of the weighting factors used to produce synthetic sensor and actuator signals (Section 7.3.2). Both designs exhibit performance which is slightly degraded in the time domain, and slightly improved in the frequency domain, relative to the Kosut two-sensor design. In contrast to the other design methods, this method deliberately enforces coupling between the residual and critical modes in order to stabilize the residual modes. As with the extended Kosut method, the relatively good performance of these designs stems largely from the existence of design parameters which can substantially influence the dynamics of (a finite number of) residual modes.

Of the design methods studied, only the Canavin and Levine-Athans methods guarantee stability in the closed-loop design model. The test design comparisons show that the assurance of design model stability, although necessary for satisfactory performance, is by no means sufficient to assure a desired level of system performance as evaluated by several classical time-domain and frequency-domain criteria.

8.3 Recommendations

The theoretical studies of Sections 3 through 7 together with the test design comparisons discussed in this section provide a rational basis for deciding where to concentrate research efforts in the near future. Specific recommendations regarding such efforts are briefly outlined below.

8.3.1 Discontinuations

The unsatisfactory test design produced by the pole assignment (Davison-Wang) method on such a simple example suggest that major theoretical advances in this method are required to make it suitable for LSS controller design. It is felt that the probability of success in such an effort is neither high enough, nor of sufficient value to the LSS control problem, to warrant continued study. We therefore propose discontinuing study of this method for the present.

8.3.2 Theoretical Studies Only

The marginal performance of the modal decoupling (Canavin) and the optimal output feedback (Levine-Athans) methods is principally due to a lack of influence of the methods upon residual mode dynamics. It is felt that theoretical study focused on an attempt to extend these methods so as to enable the designer to influence residual mode dynamics is warranted. In particular, the similarity between the Levine-Athans method and the Kosut method of minimum error excitation, together with the extensions to the Kosut methods developed in this volume (Section 6.2.3), lead us to expect that rapid and significant progress could be made with the Levine-Athans method. No large simulations with these methods are recommended at present.

8.3.3 Theoretical Development and Simulation

The excellent performance of the test designs using the suboptimal output feedback (Kosut) method, as extended, and the stochastic optimal output feedback (Johnson) method, suggest that a major effort in two (essentially parallel) directions should be undertaken:

(1) Significant theoretical developments already reported should be continued and refined. Particular focal points of interest would include, but not be limited to:

- (a) Systematic guidelines for choosing free design parameters associated with redundant sensors so as to improve system performance.
- (b) Effects of a decentralized information structure.
- (c) Development of efficient computational algorithms (Johnson).

(2) Simulations of much larger dimensional systems using these controller design methods should be undertaken.

It should be observed that these recommendations relative to the Johnson method involve relatively high risk, because of the complexity of the method, but promise quite high payoff, because of the broad scope of the method. In contrast, the recommendations relative to the Kosut methods inolve relatively low risk, because of the simplicity of the method, but promise somewhat lower payoff, because the method has narrower scope—in particular, it does not treat stochastic effects.

APPENDIX A

FUNDAMENTAL MODAL DYNAMIC MODELS OF LARGE FLEXIBLE SPACE STRUCTURES

A.1 Introduction

A flexible space structure is physically an infinite-dimensional distributed parameter system. In order that the finite-dimensional mathematical model (2-1) can satisfactorily approximate the actual structure under study, a large number (L) of generalized coordinates is required, and a large number of modes of vibration must be modeled. Theoretically if one wishes to control the vibration of such a structure, one should control all of the vibration modes. However, various practical reasons (e.g., limitations on the total weight of actuators, sensors, and control equipment, limitations on the number and type of actuators used, and limitations on the capability of onboard computers and memory) will prevent one from doing so. Feasible design of vibration control for a large flexible space structure must, therefore, be based on a finite-dimensional model of permissibly low order. On the other hand, not all vibration modes may be of equal importance to the performance of the structure, and not all modes will be equally excited. Among the excited modes, some may be of critical importance, while others have only secondary effect.

Two approaches for simplifying the finite-element model [Eq. (2-4) to (2-6)] are discussed in Sections A.2 to A.4. Methods for determining the relative importance of the vibration modes are outlined in Sections A.5 and A.6. Section A.7 contains a comment on the direct applicability of the conventional frequency-response method to large flexible space structures.

A.2 A Reduced-Order Model

Critical modes are usually of low natural frequencies, but not necessarily of the lowest ones. In the frequency spectrum, critical modes may be interspersed with residual modes. Let $[\omega_{Cj},\phi_{Cj}]$, j=1,...,N, denote the critical modes, and $[\omega_{Rk},\phi_{Rk}]$, k=1,...,M, denote the remainder of the L modeled modes. Then the finite-element modal dynamic model [Eq. (2-4) to (2-6)] can be partitioned into two parts [like Eq. (2-7) to (2-9)] as follows

$$\left\{\begin{aligned} \ddot{\eta}_C + \Omega_C^2\eta_C &= \phi_C^T B_A u \\ \ddot{\eta}_R + \Omega_R^2\eta_R &= \phi_R^T B_A u \\ y &= C_P\Phi_C\eta_C + C_V\Phi_C\dot{\eta}_C + C_P\Phi_R\eta_R + C_V\Phi_R\dot{\eta}_R \\ q &= \Phi_C\eta_C + \Phi_R\eta_R \end{aligned}\right. \tag{A-1}$$

A first and the most common approach in reducing the large model (A-1) is to completely ignore all the residual modes by assuming $\eta_R(t) \equiv 0$. What is left is the following fundamental modal design model

$$\begin{cases} \ddot{\eta}_C + \Omega_C^2 \eta_C = \Phi_C^T B_A u \\ y \simeq C_P \Phi_C \eta_C + C_V \Phi_C \dot{\eta}_C \\ q \simeq \Phi_C \eta_C \end{cases} \tag{A-2}$$

The design of control systems is then based on such a reduced model of dimension N << L.

The assumption that $\eta_R(t) \equiv 0$ may not be justified; since $\Phi_R^T B_A$ may not be identically zero, control may spill over to residual modes and significantly excite them. Thus, the control systems designed need to be evaluated first with the presence of some residual modes. For further evaluation, successful designs may then be implemented or simulated in the large-dimensional finite-element model in the presence of all modeled residual modes. See Figure 2-1.

A.3 Another Reduced-Order Model

Since flexible space structures are coupled distributed-parameter systems, the applied forces and torques may desirably influence the critical modes, but may also undesirably influence the residual modes. Among the residual modes, there may be some that, if ignored, might hamper the performance of the control systems thus designed, but, if taken into account in the design or optimization, might assist the performance. This subset of the design residual modes may, for example, include those residual modes on which undesirable influence (i.e., spillover) from the acturators is inevitable.

A second approach in reducing the large model (A-1) is to completely ignore all the residual modes except the subset described above, and to ignore only the dynamics of the latter by assuming that $\dot{\eta}_{DR}(t) \equiv 0$, where subscript "DR" denotes the indicated subset of design residual modes. Since $\dot{\eta}_{DR}(t) \equiv 0$ implies $\eta_{DR}(t) = -\Omega_{DR}^{-2} \Phi_{DR}^T B_A u(t)$, Eq. (A-1) reduces to

$$\begin{cases} \ddot{\eta}_C + \Omega_C^2 \eta_C = \Phi_C^T B_A u \\ y \simeq C_P \Phi_C \eta_C + C_V \Phi_C \dot{\eta}_C - C_P \Phi_{DR} \Omega_{DR}^{-2} \Phi_{DR}^T B_A u \\ q \simeq \Phi_C \eta_C - \Phi_{DR} \Omega_{DR}^{-2} \Phi_{DR} B_A u \end{cases} \tag{A-3}$$

Note that the fundamental modal design model remains the same as before, but vectors q and y are augmented with terms containing the input u. The design/optimization of control systems is then based on such a reduced model. The validity of assuming $\dot{\eta}_{DR}(t) \equiv 0$ and ignoring all other residual modes is open to question. Thus, the control system design still needs to be evaluated first with the presence of design residual modes, and then the evaluation residual modes. Successful designs may be implemented into the large finite-element model for further evaluation. Again see Figure 2-1.

A.4 Comments on the Reduced-Order Models

The first reduced-order model, Eq. (A-2), was used in References 1 and 2. It is similar in principle to Davison's reduced model [3], [13]. Thus, it may also suffer from having large steady-state errors in the q coordinates while providing small errors in the dynamic behavior of the critical modes, according to the historic disputes between Chidambara and Davison, [4] through [8].

Special forms of the second reduced-order model, Eq. (A-3), were considered in References 9 and 10; this model is in principle similar to Chidambara's second model [6], [14]. Thus, it may also suffer from having far different dynamic behavior in the q-coordinate while providing correct steady-state response to a specific input, according to Davison [7], [8]. Without sufficient damping on the design residual modes, such a reduced-order model cannot be developed on the basis of singular-perturbation theory as commonly understood [11], [12].

Both models will have more problems with large flexible space structures because of light damping in the systems. Moreover, the following two basic questions common to both reduced models have not been addressed before. How shall the number and the location of the actuators on the structure be selected so that all the critical modes are controllable? How shall the location of the actuators on the structure be selected so that control spillover to residual modes can be minimized? These two questions (with the equivalent ones for the sensors) must be properly answered before effective design of control systems based on either of the reduced models can be made.

A.5 Magnitude of Individual Modal Responses

The relative importance of the L vibration modes can be determined by comparing the relative magnitude of their response to expected excitations. Rewriting the discrete model (2-1) in terms of the vibration modes and modal coordinates

$$\ddot{\eta} + \Omega^2 \eta = \Phi^T f \tag{A-4}$$

and taking the Laplace transform yields

$$H(s) = \operatorname{diag}\left\{\frac{1}{s^2 + \omega_1^2}, \ldots, \frac{1}{s^2 + \omega_L^2}\right\}\left\{\Phi^T F(s) + s\eta(0) + \dot{\eta}(0)\right\} \tag{A-5}$$

where H(s) and F(s) are the Laplace transforms of the time functions $\eta(t)$ and f(t), respectively.

A.5.1 Free Vibration

With F(s) = 0, Eq. (A-5) can be rewritten in terms of initial conditions q(0) and $\dot{q}(0)$ as follows

$$H(s) = \text{diag}\left\{\frac{1}{s^2+\omega_1^2}, \ldots, \frac{1}{s^2+\omega_L^2}\right\}\phi^T M\left[sq(0) + \dot{q}(0)\right]$$

This equation is useful for analyzing free vibration of the structure subject to various initial conditions. By varying the vectors q(0) and $\dot{q}(0)$ in the equation over an expected class of initial conditions at expected locations on the structure, one can determine what modes of vibration are seriously excited most often, and hence require active control.

To a generic initial condition $(q(0), \dot{q}(0))$, the Laplace transform of the jth modal response $\eta_j(t)$ is

$$H_j(s) = \frac{1}{s^2+\omega_j^2}\,\phi_j^T M\left[sq(0) + \dot{q}(0)\right]$$

The time-domain response is thus given by

$$\eta_j(t) = \phi_j^T Mq(0)\cos\omega_j t + \phi_j^T M\dot{q}(0)\,\frac{1}{\omega_j}\sin\omega_j t$$

It describes a sinusoidal function with rms (root-mean-square) magnitude given by

$$\eta_{j\text{rms}} = 0.707\sqrt{[\phi_j^T Mq(0)]^2 + [\phi_j^T M\dot{q}(0)/\omega_j]^2}$$

Comparison of the rms magnitude of the L modal responses

$$\eta_{1\text{rms}}, \eta_{2\text{rms}}, \ldots, \eta_{L\text{rms}}$$

will determine what modes are seriously excited by the given initial excitation $(q(0), \dot{q}(0))$. Note that with everything else being unchanged, rms magnitude $\eta_{j\text{rms}}$ increases as natural frequency ω_j decreases.

A.5.2 Forced Vibration

With $q(0) = \dot{q}(0) = 0$, Eq. (A-5) becomes

$$H(s) = \text{diag}\left\{\frac{1}{s^2+\omega_1^2}, \ldots, \frac{1}{s^2+\omega_L^2}\right\}\phi^T F(s)$$

This equation is useful for analyzing forced motions of the structure subject to various external disturbances. Onboard equipment, the space environment, and maneuvers (e.g., slew) may introduce persistent, intermittent, impulsive, or random disturbances to the space structure, and hence cause it to vibrate. By varying F(s) in the equation over an expected class of disturbance forces at expected locations on the structure, one can also determine what modes are seriously excited most often, and hence require active control.

For example, consider the sinusoidal disturbances caused by onboard equipment at various locations with possibly different frequencies of vibration. A generic sinusoidal disturbance force with frequency β_k at location k can be expressed as

$$f(t) = b^k u_k(t) \tag{A-6}$$

$$u_k(t) = \sqrt{\alpha_k^2 + 1}\ \sin\left(\beta_k t + \tan^{-1}\alpha_k\right) \tag{A-7}$$

with $b^k = (b_1^k, \ldots, b_L^k)$ denoting the influence vector. Let $U_k(s)$ denote the Laplace transform of the sinusoidal disturbance input $u_k(t)$. Then

$$H(s) = \mathrm{diag}\left\{\frac{1}{s^2 + \omega_1^2}, \ldots, \frac{1}{s^2 + \omega_L^2}\right\} \phi^T b^k U_k(s)$$

Obviously, any mode (say mode j) whose frequency is equal, or sufficiently close, to some of the disturbance frequencies (say β_k) will undoubtedly be critically excited, unless the disturbance has no influence to mode j (namely, unless $\phi_j^T b^k = 0$). Any such mode, with $\phi_j^T b^k \neq 0$, must be considered a critical mode.

Consider the case where none of the natural frequencies ω_j is close to any of the disturbance frequencies β_k. The Laplace transform of response $\eta_j(t)$ is

$$H_j(s) = \frac{1}{s^2 + \omega_j^2}\ \phi_j^T b^k\ \frac{\alpha_k s + \beta_k}{s^2 + \beta_k^2} \tag{A-8}$$

By partial-fraction expansion, it becomes

$$H_j(s) = \phi_j^T b^k\ \frac{1}{-\beta_k^2 + \omega_j^2}\left[\frac{\alpha_k s + \beta_k}{s^2 + \beta_k^2} - \frac{\alpha_k s + \beta_k}{s^2 + \omega_j^2}\right]$$

Hence, the time-domain response $\eta_j(t)$ is

$$\eta_j(t) = \phi_j^T b^k \frac{1}{-\beta_k^2 + \omega_j^2} \left\{ \sqrt{\alpha_k^2 + 1} \sin [\beta_k t + \tan^{-1} \alpha_k] \right.$$

$$\left. - \sqrt{\alpha_k^2 + \beta_k^2/\omega_j^2} \sin [\omega_j t + \tan^{-1} (\alpha_k \omega_j/\beta_k)] \right\} \qquad \text{(A-9)}$$

The rms magnitude of this periodic function is

$$\eta_{j\mathrm{rms}} = 0.707 \left| \phi_j^T b^k \frac{1}{-\beta_k^2 + \omega_j^2} \right| \sqrt{2\alpha_k^2 + 1 + \beta_k^2/\omega_j^2} \qquad \text{(A-10)}$$

Similarly, comparison of these rms magnitudes determines what modes are seriously excited by the sinusoidal vibration of the onboard equipment at location k. Note that, with influence $|\phi_j^T b^k|$ being the same for all modes, the rms magnitude $\eta_{j\mathrm{rms}}$ increases as the natural frequency ω_j approaches the disturbance frequency β_k, or as the natural frequency decreases.

It is worth mentioning that if one followed the usual frequency-response method [15]-[20] for undamped systems, one would erroneously obtain the rms magnitude as

$$\eta_{j\mathrm{rms}} = 0.707 \left| \phi_j^T b^k \frac{1}{-\beta_k^2 + \omega_j^2} \right| \sqrt{\alpha_k^2 + 1} \qquad \text{(A-11)}$$

missing the term $\alpha_k^2 + \beta_k^2/\omega_j^2$ under the radical sign. See Eq. (A-20) of Section A.7.1. The difference

$$\sqrt{2\alpha_k^2 + 1 + \beta_k^2/\omega_j^2} - \sqrt{\alpha_k^2 + 1}$$

can result in a significant error in the comparison of the L rms magnitudes, since it varies with the vibration modes. Moreover, if one assesses the effect of the disturbance by actually measuring the rms magnitude of response $\eta_j(t)$, one may over-assess the effect, because the measurements correspond to Eq. (A-10), rather than Eq. (A-11) as conventionally used.

A.6 Individual Contributions to Line-of-Sight Error

The relative importance of the vibration modes can also be determined by comparing their individual contributions to a given performance index on the attitude or shape of the structure, such as the line-of-sight error. The Laplace transform Q(s) of the response q(t) can be obtained from Eq. (A-5) as

$$Q(s) = \Phi H(s)$$

$$= \sum_{j=1}^{L} \frac{1}{s^2 + \omega_j^2} \phi_j \{\phi_j^T F(s) + s\eta_j(0) + \dot{\eta}_j(0)\} \qquad \text{(A-12)}$$

A.6.1 Free Vibration

With $F(s) \equiv 0$, Eq. (A-12) becomes

$$Q(s) = \sum_{j=1}^{L} \frac{s\eta_j(0) + \dot{\eta}_j(0)}{s^2 + \omega_j^2} \phi_j$$

This equation is useful for analyzing the effect on the attitude and shape of the structure when a specific class of vibration modes is initially excited. Given a performance index, the relative importance of these modes can then be assessed.

For example, consider the line-of-sight error which is expressed as a linear combination of the generalized coordinates as follows

$$e = c^T q \triangleq c_1 q_1 + \ldots + c_L q_L \qquad \text{(A-13)}$$

where $c = (c_1, \ldots, c_L)$ is the L-vector of constant coefficients. Its Laplace transform is readily given by

$$E(s) = c^T Q(s) = \sum_{j=1}^{L} c^T \phi_j \frac{s\eta_j(0) + \dot{\eta}_j(0)}{s^2 + \omega_j^2}$$

Consequently, the rms magnitude of the line-of-sight error is given by

$$e_{rms} = 0.707 \sum_{j=1}^{L} |c^T \phi_j| \sqrt{\eta_j^2(0) + \dot{\eta}_j^2(0)/\omega_j^2}$$

where the term

$$e_{jrms} \triangleq 0.707 |c^T \phi_j| \sqrt{\eta_j^2(0) + \dot{\eta}_j^2(0)/\omega_j^2}$$

represents the rms contribution from mode j. Comparison of the individual rms contributions $e_{1rms}, \ldots, e_{Lrms}$ determines what modes of vibration are critical to the line-of-sight accuracy.

A.6.2 Forced Vibration

With $q(0) = \dot{q}(0) = 0$, Eq. (A-12) becomes

$$Q(s) = \sum_{j=1}^{L} \frac{1}{s^2 + \omega_j^2} \phi_j \phi_j^T F(s)$$

This equation is useful for analyzing the effect on the attitude and shape of the structure caused by various disturbances (e.g., onboard equipment, the space environment, structural maneuvers) and by the vibration modes thereby excited. Given a performance index, the relative importance of the natural modes and the necessity of compensating the disturbance can be assessed.

Consider again the line-of-sight error defined by Eq. (A-13). Assume onboard equipment at location k produces a sinusoidal disturbance with frequency β_k and phase angle $\tan^{-1} \alpha_k$, as expressed by Eq. (A-6) and (A-7). Then, the Laplace transform of the line-of-sight error is

$$E(s) = c^T Q(s) = \sum_{j=1}^{L} \frac{1}{s^2 + \omega_j^2} c^T \phi_j \phi_j^T b^k \frac{\alpha_k s + \beta_k}{s^2 + \beta_k^2} \qquad \text{(A-14)}$$

Again, any mode (say mode j) whose natural frequency is equal, or sufficiently close, to the disturbance frequency β_k will be critically excited. Such a natural mode, and such a sinusoidal disturbance, will be critical to the line-of-sight accuracy, unless it contributes nothing to the line-of-sight error (namely, unless $c^T \phi_j \phi_j^T b^k = 0$).

Now, assume no ω_j is close to β_k. As with Eq. (A-9) - (A-10), the time-domain line-of-sight error and its rms magnitude are given by

$$e(t) = \sum_{j=1}^{L} c^T\phi_j\phi_j^T b^k \frac{1}{-\beta_k^2 + \omega_j^2}\left\{\sqrt{\alpha_k^2 + 1}\ \sin\ [\beta_k t + \tan^{-1}\ \alpha_k]\right.$$

$$\left. - \sqrt{\alpha_k^2 + \beta_k^2/\omega_j^2}\ \sin\ [\omega_j t + \tan^{-1}\ (\alpha_k\omega_j/\beta_k)]\right\}$$

$$= \left[\sum_{j=1}^{L} c^T\phi_j\phi_j^T b^k \frac{1}{-\beta_k^2 + \omega_j^2}\right]\sqrt{\alpha_k^2 + 1}\ \sin\ [\beta_k t + \tan^{-1}\ \alpha_k]$$

$$- \sum_{j=1}^{L}\left\{c^T\phi_j\phi_j^T b^k \frac{1}{-\beta_k^2 + \omega_j^2}\sqrt{\alpha_k^2 + \beta_k^2/\omega_j^2}\right\}\sin\ [\omega_j t + \tan^{-1}\ (\alpha_k\omega_j/\beta_k)] \qquad \text{(A-15)}$$

$$e_{rms} = 0.707\left\{\left[\sum_{j=1}^{L} c^T\phi_j\phi_j^T b^k \frac{1}{-\beta_k^2 + \omega_j^2}\right]^2 (\alpha_k^2 + 1)\right.$$

$$\left. + \sum_{j=1}^{L}\left[c^T\phi_j\phi_j^T b^k \frac{1}{-\beta_k^2 + \omega_j^2}\right]^2 (\alpha_k^2 + \beta_k^2/\omega_j^2)\right\}^{1/2} \qquad \text{(A-16)}$$

Notice the presence of the terms

$$e_{jrms} \triangleq 0.707\left|c^T\phi_j\phi_j^T b^k \frac{1}{-\beta_k^2 + \omega_j^2}\right|\sqrt{\alpha_k^2 + \beta_k^2/\omega_j^2}\ ,\ j = 1,\ldots,L \qquad \text{(A-17)}$$

Each of these terms represents the rms contribution from an individual natural mode of vibration to the overall line-of-sight error. Comparison of these contritutions e_{1rms} ,..., e_{Lrms} will determine what modes of vibration are important to the line-of-sight accuracy.

Notice also that the contribution from the disturbance to the line-of-sight error [Eq. (A-16)] is the absolute value of the algebraic sum of L terms

$$\left|\sum_{j=1}^{L} c^T\phi_j\phi_j^T b^k \frac{1}{-\beta_k^2 + \omega_j^2}\right|\sqrt{\alpha_k^2 + 1}$$

This contribution may be large or small, depending on whether these L terms actually add up or cancel themselves out. Its comparison with the individual contributions e_{1rms} ,..., e_{Lrms} from the natural vibration modes will determine whether or not the disturbance is important enough to require compensation.

Again, for such an undamped system, if one followed the conventional frequency-response method ([15] through [20]), or equivalently, the phasor method ([17],[18], [20]), one would have erroneously obtained the rms line-of-sight error as

$$e_{rms} = 0.707 \left| \sum_{j=1}^{L} c^T \phi_j \phi_j^T b^k \frac{1}{-\beta_k^2 + \omega_j^2} \right| \sqrt{\alpha_k^2 + 1} \qquad \text{(A-18)}$$

(see Eq. (A-21) of Section A.7.1.) The L terms representing the contribution from the vibration modes would have been missed. As a result, the line-of-sight error would be significantly underestimated.

A.7 Steady-State Sinusoidal Response

A.7.1 Frequency-Response Method; Phasor Method

The approach, known as the frequency-response method ([15] through [19]) or the phasor method ([17], [18], and [20]), is a convenient tool for finding the steady-state response to sinusoidal inputs. Given the transfer function T(s) between the response Y(s) and the input U(s), and a sinusoidal input u(t) with frequency ω, the amplitude of the steady-state response y(t) is given by

$$y_{amp} = |T(i\omega)| u_{amp}$$

where $|T(i\omega)|$ denotes the absolute value of the transfer function T(s), with s replaced by $i\omega$, and where u_{amp} denotes the amplitude of u(t). Hence, the rms magnitude of the response y(t) is given by

$$y_{rms} = |T(i\omega)| u_{rms} \qquad \text{(A-19)}$$

where u_{rms} denotes the rms magnitude of input u(t).

Consider, for example, the modal response $\eta_j(t)$ to the sinusoidal disturbance input $u_k(t)$ described by Eq. (A-7). The transfer function T(s) can be obtained from Eq. (A-8) as

$$T(s) = H_j(s)/U_k(s) = \frac{1}{s^2 + \omega_j^2} \phi_j^T b^k$$

Setting $s = i\beta_k$ yields

$$\left|T(i\beta_k)\right| = \left|\frac{1}{-\beta_k^2 + \omega_j^2} \phi_j^T b^k\right|$$

where β_k is the disturbance input frequency. The rms magnitude of the disturbance input $u_k(t)$ is $0.707\sqrt{\alpha_k^2 + 1}$. Therefore, by Eq. (A-19), the rms magnitude of the modal response $\eta_j(t)$ is given by

$$\eta_{jrms} = \left|\frac{1}{-\beta_k^2 + \omega_j^2} \phi_j^T b^k\right| 0.707 \sqrt{\alpha_k^2 + 1} \tag{A-20}$$

as was given in Section A.5.2 as Eq. (A-11).

Consider for another example the line-of-sight error due to the same sinusoidal input. From Eq. (A-14), the transfer function T(s) is obtained as

$$T(s) = E(s)/U_k(s) = E(s)\Big/\frac{\alpha_k s + \beta_k}{s^2 + \beta_k^2}$$

$$= \sum_{j=1}^{L} \frac{1}{s^2 + \omega_j^2} c^T \phi_j \phi_j^T b^k$$

Consequently, the rms magnitude of the line-of-sight error is given by

$$e_{rms} = \left|\sum_{j=1}^{L} \frac{1}{-\beta_k^2 + \omega_j^2} c^T \phi_j \phi_j^T b^k\right| \sqrt{\alpha_k^2 + 1} \tag{A-21}$$

as was given in Section A.6.2 as Eq. (A-18).

A.7.2 <u>Commentary</u>

The frequency-response method (equivalently, the phasor method) is actually valid only when the steady-state response is a sinusoidal function having exactly the same frequency as the input. It requires that each of the system modes die out after a sufficient amount of time has elapsed. This method has been very successful and useful in the past because most engineering systems encountered in practice have sufficient damping on all modes. However, it may not be directly applicable to future large flexible space structures, which will have very little damping, and which are modeled (and simulated by the computer) as undamped systems of harmonic oscillators. Vibration modes of

such structures will never die out once excited; they all become parts of the steady-state response. Thus, special care must be taken in order to avoid drawing incorrect conclusions from the steady-state response of undamped systems.

Undamped systems traditionally have been used to illustrate various principles of system analysis and synthesis. Nevertheless, they were not as carefully studied as systems with damping, because they are not realistic systems. Some conclusions on system design may be wrong in the context of undamped systems.

References

1. Balas, M.J., "Active Control of Flexible Systems," Proceedings of AIAA Symposium on Dynamics and Control of Large Flexible Spacecraft, Blacksburg, VA, June 1977.

2. Canavin, J.R., "The Control of Spacecraft Vibration Using Multivariable Output Feedback," Proceedings of AIAA/AAS Astrodynamics Conference, Palo Alto, CA, Aug. 1978, Paper 78-1419.

3. Davison, E.J., "A Method for Simplifying Linear Dynamic Systems," IEEE Transactions on Automatic Control, Vol. AC-11, No. 1, pp. 93-101, Jan. 1966.

4. Chidambara, M.R., "On 'A Method for Simplifying Linear Dynamic Systems,'" IEEE Transactions on Automatic Control, Vol. AC-12, pp. 119-120, Feb. 1967.

5. Davison, E.J. "Author's Reply," Ibid., pp. 120-121.

6. Chidambara, M.R. "Further Remarks on Simplifying Linear Dynamic Systems," Ibid., pp. 213-214, April 1967.

7. Davison, E.J., "Author's Reply," Ibid., pp. 214.

8. Chidambara, M.R., and E.J. Davison, "Further Comments on 'A Method for Simplifying Linear Dynamic Systems,'" "Reply," "Further Comments," "Further Reply," Ibid., Vol. AC-12, pp. 799-800, Dec. 1967.

9. Sesak, J.R., "Control of Large Space Structures via Singular Perturbation Optimal Control," AIAA Conf. on Large Space Platforms: Future Needs and Capabilities, Los Angeles, CA, Sept. 1978, paper 78-1690.

10. Gran, R., and M. Rossi, "Large Space Structures Control: What are the Problems? What are the Solutions?" Ibid., paper 78-1688.

11. Kokotovic, P.V., "A Control Engineer's Introduction to Singular Perturbations," in Singular Perturbations: Order Reduction in Control Systems, pp. 1-12, ASME, New York, 1972.

12. Kokotovic, P.V., R.E. O'Malley, Jr., and P. Sannuti, "Singular Perturbations and Order Reduction in Control Theory—An Overview," Automatica, Vol. 12, pp. 123-132, March 1976.

13. Mahapatra, G.B., "A Note on Selecting Low-Order Systems by Davison's Model Simplification Technique," IEEE Transactions on Automatic Control, Vol. AC-22, pp. 677-678, August 1977.

14. Mahapatra, G.B., "A Further Note on Selecting a Low-Order System Using the Dominant Eigenvalue Concept," IEEE Transactions on Automatic Control, Vol. 24, No. 1, pp. 135-136, Feb 1979.

References (Cont.)

15. Meirovitch, L., Elements of Vibration Analysis, New York: McGraw-Hill, 1975.

16. Hurty, W.C., and M.F. Rubinstein, Dynamics of Structures, Englewood Cliffs, N.J.: Prentice-Hall, 1964.

17. Eveleigh, V.W., Introduction to Control Systems Design, New York: McGraw-Hill, 1972.

18. D'Azzo, J.J., and C.H. Houpis, Linear Control System Analysis and Design: Conventional and Modern, New York: McGraw-Hill, 1975.

19. Dorf, R.C., Modern Control Systems, Reading, Mass.: Addison-Wesley, 1967.

20. Nilsson, J.W., Introduction to Circuits, Instruments and Electronics, New York: Harcourt, Brace & World, 1968.

APPENDIX B

CONTROLLABILITY AND OBSERVABILITY OF VIBRATION MODES

B.1 Introduction

Consider the problem of controlling the vibration modes of an undamped large flexible space structure [1], [2]. The dynamics of such a structure are usually approximated by the method of finite elements (using computer programs such as NASTRAN, STRDYNE) as system (2.1). Combining with equations (2-2) and (2-3) for control inputs and measurement outputs yield

$$\begin{cases} M\ddot{q} + Kq = B_A u \\ y = C_P q + C_V \dot{q} \end{cases} \tag{B-1}$$

Expressed in terms of natural modes $\{\omega_j, \phi_j\}$ of vibration and modal coordinates η_j, it becomes

$$\begin{cases} \ddot{\eta} + \Omega^2 \eta = \Phi^T B_A u \\ y = C_P \Phi \eta + C_V \Phi \dot{\eta} \\ q = \Phi \eta \end{cases} \tag{B-2}$$

See Section 2.2.2 for a description of the notations. It follows from system (B-2) that for mode (ω_j, ϕ_j) to be controllable by input u, the location of the actuators must be such that the jth row of the influence matrix $\Phi^T B_A$ is nonzero; i.e., $\phi_j^T B_A \neq 0$. For a [illegible], (ω_j, ϕ_j) and (ω_k, ϕ_k), to be controllable, one would expect the same; namely, that both row vectors $\phi_j^T B_A$ and $\phi_k^T B_A$ are nonzero. It is not obvious, however, that one should require more on the location and the number of actuators if two or more modes have a common natural frequency. The phenomenon of common frequencies is not unusual in the case of symmetric structures.

The situation regarding the location and number of sensors required for observability of critical modes is similar, as expected by the duality between observation and control.

Section B.2 first describes two different directions from which the design of structural vibration control systems can be approached, then gives a brief review of the notion of controllability and observability. Because of duality, subsequent discussions concentrate on controllability. Sections B.3 through B.7 spell out various necessary and sufficient conditions for

complete controllability of critical modes under various circumstances. An algorithm embracing all these controllability conditions is proposed in Section B.8 for systematically determining the proper location and minimum number of actuators required for ensuring at least complete controllability of critical modes.

B.2 Alteration of Modes of Response, Regulation of Modal Responses; Complete Controllability, Complete Observability

The design of control systems for suppressing structural vibration can be approached from two different directions: alteration of modes of response in the spirit of frequency-domain modal control theory and regulation of modal responses in the spirit of time-domain state (or output) control theory.

B.2.1 Alteration of Modes of Response

Modal control of a multivariable linear time-invariant system is, by definition [3] - [5], to alter the modes of system response to achieve the desired control objectives. The modes of the system response are characterized by the poles of the system. Without redesigning the given system, a common approach is to introduce appropriate feedback so that the closed-loop system has the desired poles, hence the desired modes of response. However, one can only alter the characteristics of the completely controllable and completely observable part [5] of the open-loop system. Therefore, the location of the actuators should be such that all those response modes which one wishes to alter are controllable. Similarly, the location of the sensors should be such that all modes to be measured are observable.

A large flexible space structure generally has a large number of vibration modes [1], [2], while the feasible number of sensors and especially that of actuators placed on the structure for effecting the desired alteration are relatively small. Moreover, because of practical limitations on the design of feedback loops, one can actually alter a very limited number of vibration modes. Consequently, one must concentrate on those vibration modes that are fundamentally important to the performance of the structure. See Appendix A for discussions of critical modes of vibration and their determination.

Partitioned into the critical and the modeled residual parts, the modal model (B-2) becomes

$$\begin{cases} \ddot{\eta}_C + \Omega_C^2 \eta_C = \Phi_C^T B_A u \\ \ddot{\eta}_R + \Omega_R^2 \eta_R = \Phi_R^T B_A u \\ y = C_P \Phi_C \eta_C + C_V \Phi_C \dot{\eta}_C + C_P \Phi_R \eta_R + C_V \Phi_R \dot{\eta}_R \\ q = \Phi_C \eta_C + \Phi_R \eta_R \end{cases} \tag{B-3}$$

See Section 2.2.3 for a description of the notation. A corresponding state-space representation of system (B-3) is

$$
\begin{bmatrix} \dot{\eta}_C \\ \ddot{\eta}_C \\ \hline \dot{\eta}_R \\ \ddot{\eta}_R \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & I & & \\ -\Omega_C^2 & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & I \\ \multicolumn{2}{c|}{0} & -\Omega_R^2 & 0 \end{array}\right] \begin{bmatrix} \eta_C \\ \dot{\eta}_C \\ \hline \eta_R \\ \dot{\eta}_R \end{bmatrix} + \begin{bmatrix} 0 \\ \Phi_C^T B_A \\ \hline 0 \\ \Phi_R^T B_A \end{bmatrix} u
$$

$$
y = [C_P\Phi_C \;\; C_V\Phi_C \,|\, C_P\Phi_R \;\; C_V\Phi_R] \begin{bmatrix} \eta_C \\ \dot{\eta}_C \\ \hline \eta_R \\ \dot{\eta}_R \end{bmatrix} \tag{B-4}
$$

where I denotes the identity matrix of appropriate dimension. The 2L-vector $(\eta_C, \dot{\eta}_C, \eta_R, \dot{\eta}_R)$ represents the state of the 2-L dimensional linear system (B-4). To be able to alter the critical modes of the system response, one must make the following part:

$$
\left\{ \begin{aligned} \begin{bmatrix} \dot{\eta}_C \\ \ddot{\eta}_C \end{bmatrix} &= \begin{bmatrix} 0 & I \\ -\Omega_C^2 & 0 \end{bmatrix} \begin{bmatrix} \eta_C \\ \dot{\eta}_C \end{bmatrix} + \begin{bmatrix} 0 \\ \Phi_C^T B_A \end{bmatrix} u \\ y &= [C_P\Phi_C \;\; C_V\Phi_C] \begin{bmatrix} \eta_C \\ \dot{\eta}_C \end{bmatrix} \end{aligned} \right. \tag{B-5}
$$

completely controllable and completely observable. Since this fundamental system (B-5) has 2N poles (i.e., N pairs of conjugate imaginary poles), for altering the modes of its response, a set of 2N desired closed-loop poles may be chosen to replace them. See Section B.2.4.

B.2.2 Regulation of Modal Responses

The modal model (Eq. (B-2)) can be recast into the following state-variable form

$$
\left\{ \begin{aligned} \begin{bmatrix} \dot{\eta} \\ \ddot{\eta} \end{bmatrix} &= \begin{bmatrix} 0 & I \\ -\Omega^2 & 0 \end{bmatrix} \begin{bmatrix} \eta \\ \dot{\eta} \end{bmatrix} + \begin{bmatrix} 0 \\ \Phi^T B_A \end{bmatrix} u \\ y &= [C_P\Phi \;\; C_V\Phi] \begin{bmatrix} \eta \\ \dot{\eta} \end{bmatrix} \end{aligned} \right. \tag{B-6}
$$

with $\eta_1, \ldots, \eta_L, \dot{\eta}_1, \ldots, \dot{\eta}_L$ denoting the 2L state variables. Regulation is, by definition [6], [7], to reduce modal response from any initial state $(\eta(0), \dot{\eta}(0))$ to zero directly, and to maintain an optimally small total error, in contrast to the alteration of response modes. A feedback regulator usually requires a state regulator with a state estimator (a Luenberger observer or a Kalman filter). Alternatively, an output feedback controller with or without dynamic compensators [8] may be required.

However, the regulation of modal responses of a large space structure, like the alteration of modes of structural response, should concentrate on fundamentally important modes of vibration. Partitioned in terms of critical and modeled residual modes, system (B-6) can be rewritten as system (B-4), whose fundamental subsystem is given by (B-5). One should therefore concentrate on regulating the 2N-dimensional state vector $(\eta_C, \dot{\eta}_C)$ of system (B-5), since N<<L. Complete controllability and complete observability of the system to be regulated are prerequisite to satisfactory regulation.

B.2.3 Complete Controllability and Complete Observability; Duality

Before defining the controllability and observability of critical modes in Section B.3, recall the definition of controllability and observability of state [9]. To do so, consider the following standard state-space representation of linear time-invariant systems

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{B-7}$$

where $x \equiv (x_1, \ldots, x_n)$ denotes a vector of n state variables, $u \equiv (u_1, \ldots, u_m)$ a vector of m inputs, and $y \equiv (y_1, \ldots, y_\ell)$ a vector of ℓ outputs.

An initial state x^0 is said to be controllable if it can be transferred to a zero state in a finite length of time by some control input u(t). If every initial state is controllable, the system (B-7) is said to be completely controllable. An initial state x^0 is said to be observable if it can be determined from the knowledge of zero-input observation output y(t) for a finite length of time. If every initial state is observable, the system is said to be completely observable. See References 6, and 9 through 11.

The following are well known criteria for complete controllability and complete observability of state [5], [10], [11]. System (B-7) is completely controllable if and only if the n×mn matrix

$$Q \triangleq [B, AB, \ldots, A^{n-1}B] \tag{B-8}$$

has rank n. It is completely observable if and only if the $\ell n \times n$ matrix

$$P \triangleq \begin{bmatrix} C \\ AC \\ \vdots \\ A^{n-1}C \end{bmatrix} \tag{B-9}$$

has rank n.

The following alternative definition and interpretation given in Reference 5 is useful in understanding why complete controllability and complete observability are related to the possibility of altering system characteristics (i.e., modes of response) by feedback loops.

System (B-7) is said to be <u>completely controllable</u> if it is not algebraically equivalent, for all $t \geq 0$, to a system of the type

$$\begin{cases} \dot{x}^1 = A^{11}x^1 + A^{12}x^2 + B^1u \\ \dot{x}^2 = A^{22}x^2 \\ y = C^1x^1 + C^2x^2 \end{cases} \tag{B-10}$$

where x^1 and x^2 are vectors of n_1 and $n_2 = n - n_1$ components, respectively. In other words, it is not possible to find a coordinate system in which the state variables $x \equiv (x_1, \ldots, x_n)$ are separated into two groups, $x^1 \equiv (x^1_1, \ldots, x^1_{n_1})$ and $x^2 \equiv (x^2_1, \ldots, x^2_{n_2})$, such that the second group <u>is not affected</u> by either the first group or by the inputs to the system.

Similarly, system (B-7) is said to be <u>completely observable</u> if it is not algebraically equivalent, for all $t \leq 0$, to a system of the type

$$\begin{cases} \dot{x}^1 = A^{11}x^1 + B^1u \\ \dot{x}^2 = A^{21}x^1 + A^{22}x^2 + B^2u \\ y = C^1x^1 \end{cases} \tag{B-11}$$

where x^1 is an n_1-vector and x^2 an $(n - n_1)$-vector. In other words, it is not possible to find a coordinate system in which the state variables are separated into two groups, such that the second group <u>does not affect</u> either the first group or the outputs of the system.

It is now well known (e.g., see [5], [9], [11], [12]) that for every conclusion concerning controllability, there is a corresponding one concerning observability, and vice versa. Therefore, subsequent discussions are focused on the controllability of critical modes.

A strong relationship between complete controllability of an open-loop system and the possibility that the system has desired closed-loop poles by means of feedback was given by Wonham [13] as follows. System (B-7) is completely controllable if and only if, for every choice of n real or complex-conjugate numbers, there is a feedback matrix F such that the closed-loop system matrix A+BF has these n numbers for its eigenvalues.

B.3 Controllability of Critical Modes

In the sequel, we say that the N critical modes of structural vibration are completely controllable if and only if the (open-loop) fundamental system (B-5) is completely controllable. It follows from the best known controllability criterion (Eq. (B-8)) that the N critical modes are completely controllable if and only if the 2N×(2mN) matrix

$$Q_{2N} \triangleq [B_C, A_CB_C, \ldots, A_C^{2N-1}B_C] \tag{B-12}$$

has rank 2N, where

$$A_C = \begin{bmatrix} 0 & I \\ -\Omega_C^2 & 0 \end{bmatrix}, \quad B_C = \begin{bmatrix} 0 \\ \Phi_C^T B_A \end{bmatrix} \tag{B-13}$$

After some matrix manipulations, a simplified form of this criterion can be given.

Theorem 1: The N critical modes are completely controllable if and only if the N×(mN) matrix

$$Q_N \triangleq [\Phi_C^T B_A, -\Omega_C^2 \Phi_C^T B_A, \ldots, (-\Omega_C^2)^{N-1} \Phi_C^T B_A] \tag{B-14}$$

has rank N.

Proof: By induction, it is easy to show that

$$A_C^{2i} B_C = \begin{bmatrix} (-\Omega_C^2)^i \, \Phi_C^T B_A \\ 0 \end{bmatrix}, \quad A_C^{2i+1} B_C = \begin{bmatrix} 0 \\ (-\Omega_C^2)^i \, \Phi_C^T B_A \end{bmatrix}$$

for $i = 0, 1, 2, \ldots$. The matrix Q_{2N} therefore becomes

$$Q_{2N} = \begin{bmatrix} 0 & B_{2C} & 0 & -\Omega_C^2 B_{2C} & \cdots & 0 & (-\Omega_C^2)^{N-1} B_{2C} \\ B_{2C} & 0 & -\Omega_C^2 B_{2C} & 0 & \cdots & (-\Omega_C^2)^{N-1} B_{2C} & 0 \end{bmatrix}$$

where B_{2C} denotes $\phi_C^T B_A$. Rearranging the columns, an equivalent form of Q_{2N} is given by

$$Q_{2N} = \begin{bmatrix} Q_N & 0 \\ 0 & Q_N \end{bmatrix}$$

where Q_N is as defined by Eq. (B-14). Matrix Q_{2N} has rank 2N if and only if matrix Q_N has rank N. This proves the theorem.

A statement similar to Theorem 1 was given in Reference 1 for the class of flexible structures describable by a generalized wave equation. A simple and direct consequence of Theorem 1 is the following sufficient condition.

Corollary 1: For the N critical modes to be completely controllable, it is sufficient that the influence matrix $\phi_C^T B_A$ have rank N.

To satisfy this sufficient condition requires a large number of specially placed actuators. First of all, for the N×m influence matrix to have rank N, it is a prerequisite that $m \geq N$; namely, that there are at least as many actuators on the structure as there are critical modes of vibration. Secondly, these actuators must be so located (and distributed) that the N row vectors $\phi_{C1}^T B_A, \ldots, \phi_{CN}^T B_A$ are not only nonzero but also linearly independent. This means that each one of the N critical modes is controlled independently of any other critical mode by at least one actuator. It is just a sufficient condition, but a very restrictive one. In some cases, it may be enough to use only one actuator, as was also observed in Reference 1. The following states precisely the necessary and sufficient conditions for using only one actuator.

Theorem 2: Assume only one actuator is used (i.e., m = 1). Let b_A denote its N-dimensional influence vector (i.e., $B_A = b_A$). Then the N critical modes are completely controllable by a single actuator if and only if

(1) All natural frequencies of the critical modes are distinct.

(2) The influence vector b_A is not orthogonal to any of the critical mode shapes; i.e.,

$\phi_{Cj}^T b_A \neq 0$ for all $j = 1, \ldots, N$.

Proof: Since $\phi_{Cj}^T b_A$ are scalars, the matrix Q_N of Theorem 1 simplifies to

$$Q_N = \begin{bmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_N \end{bmatrix} \begin{bmatrix} 1 & -\omega_{C1}^2 & \cdots & (-\omega_{C1}^2)^{N-1} \\ \vdots & \vdots & & \vdots \\ 1 & -\omega_{CN}^2 & \cdots & (-\omega_{CN}^2)^{N-1} \end{bmatrix}$$

where $b_j = \phi_{Cj}^T b_A$. The second square matrix on the right-hand side is a Vandermonde matrix, whose determinant is nonzero if and only if $\omega_{C1}, \ldots, \omega_{CN}$ are distinct. Therefore, matrix Q_N has rank N if and only if both conditions (1) and (2) are satisfied. In view of Theorem 1, this theorem is now proved.

Condition (2) is evident from Eq. (B-2), at least intuitively, since the actuator must have nonzero influence on each of the critical modes to be able to control all of them. But it is not as obvious that condition (2) by itself is not sufficient. Condition (1) implies that, in general, one actuator is not enough, specifically when two or more critical modes have identical or nearly identical natural frequencies.

B.4 Controllability of Critical Modes Having Identical Natural Frequencies

To apply Theorem 1, it is convenient to recognize that the matrix Q_N is exactly the controllability matrix of the following hypothetical* N-dimensional dynamic system

$$\dot{z} = -\Omega_C^2 z + \phi_C^T B_A u \qquad \text{(B-15)}$$

It follows from Theorem 1 that complete controllability of the N critical modes can be interpreted as complete controllability of the hypothetical system (B-15), and vice versa. The "conveniences" are the diagonal system matrix and the real eigenvalues. The system has only pure exponential decay modes. Moreover, the system matrix is already in the Jordan canonical form with all Jordan blocks being 1×1. Consequently, there are as many identical Jordan blocks as there are identical natural frequencies among the critical modes. With the aid of this simple hypothetical system (B-15) and the decomposition (B-10), the following results are obtained.

Theorem 3: The N critical modes are completely controllable if and only if

(1) Each row of the influence matrix $\phi_C^T B_A$ is nonzero.

(2) All those rows of matrix $\phi_C^T B_A$ which correspond to a repeated natural frequency are linearly independent.

---

* System (B-15) is not a legitimate dynamic subsystem of (B-5) by definition, and does not really exist.

Proof: Direct proof by algebraic manipulations on the matrix Q_N of Theorem 1 is not difficult, but the following is more intuitively appealing. The necessity of condition (1) is trivial. To prove that the addition of condition (2) to condition (1) is necessary, assume that the natural frequency ω_{Cj} is repeated r times among the fundamental modes. Let $\phi_j, \phi_{j+1}, \ldots, \phi_{j+r}$ denote the corresponding mode shapes. Then the corresponding equations in system (B-15) are

$$\begin{cases} \dot{z}_j = -\omega_{Cj}^2 z_j + \phi_j^T B_A u \\ \dot{z}_{j+1} = -\omega_{Cj}^2 z_{j+1} + \phi_{j+1}^T B_A u \\ \vdots \\ \dot{z}_{j+r} = -\omega_{Cj}^2 z_{j+r} + \phi_{j+r}^T B_A u \end{cases} \tag{B-16}$$

Now suppose on the contrary that the corresponding rows $\phi_j^T B_A$, $\phi_{j+1}^T B_A$, ..., $\phi_{j+r}^T B_A$ of matrix $\phi_C^T B_A$ are not linearly independent. Then

$$c_1 \phi_j^T B_A + c_2 \phi_{j+1}^T B_A + \ldots + c_{r+1} \phi_{j+r}^T B_A = 0$$

for some constants $c_1, c_2, \ldots, c_{r+1}$, not all zero. Making the corresponding combination of the r + 1 equations in system (B-16) yields

$$\dot{z}_{jc} = -\omega_{Cj}^2 z_{jc} \tag{B-17}$$

where

$$z_{jc} \triangleq c_1 z_j + c_2 z_{j+1} \cdots + c_{r+1} z_{j+r}$$

denotes a combined state variable. Replacing the last equation in system (B-16) by Eq. (B-17), the form of system (B-10) is reached. By the alternative definition of complete controllability given there, these r + 1 critical modes are not completely controllable. Therefore, the necessity of condition (2) is proven. The arguments for the sufficiency of conditions (1) and (2) combined are similar, but converse, to those for their necessity. With both conditions (1) and (2) satisfied, system (B-15) cannot be decomposed or transformed to contain any equation of the form of (B-17), and hence is completely controllable.

The following are two simple, but useful, corollaries of Theorem 3. Corollary 2 is essentially the same as Theorem 2 while Corollary 3 is just the opposite.

Corollary 2: If all natural frequencies of the critical modes are distinct and if the influence matrix $\phi_C^T B_A$ has at least one column whose N elements are all nonzero, then the N critical modes are completely controllable.

Corollary 3: For the N critical modes to be completely controllable, the number of actuators used on the structure must be at least equal to the maximum multiplicity of the natural frequencies over the critical modes. In other words, if $\mu(\omega_{Cj})$ denotes the number of critical modes having the same frequency ω_{Cj}, then it is necessary that

$$m \geq \max_{1 \leq j \leq N} \mu(\omega_{Cj}).$$

Corollary 3 means that whenever there are two or more critical modes having identical natural frequencies, a single input cannot control all the critical modes. In other words, relocation of the single actuator will not work; nor will any combination of the multiple actuators on the structure if all of them are still driven by a common input.

B.5 Controllability of Critical Modes Having Natural Frequencies Identical to Residual Modes

If some critical modes and some residual modes have a common natural frequency, the preceding conclusions must be modified except for a very special case.

Consider the special case first. This occurs when all those residual modes having natural frequencies identical to some critical modes are not influenced by the actuators (i.e., $\phi_{Rk}^T B_A = 0$). Then, all the preceding conclusions (Theorems 1-3, Corollaries 1-3) are valid without modification. In other words, such residual modes are still as ignorable as the others.

Now consider the general case. Call a residual mode (ω_{Rk}, ϕ_{Rk}) an associated residual mode if its natural frequency is identical to some critical mode and if it has nonzero influence from the actuators (i.e., $\phi_{Rk}^T B_A \neq 0$). Associated residual modes are not ignorable so far as the complete controllability of the critical modes is concerned. The number and location of the actuators required may thereby be affected.

Let matrices Ω_C^2 and Φ_C be augmented to include all the associated residual modes as if they were additional critical modes. Let $\Omega_{C'}^2$ and $\Phi_{C'}$ denote the augmented matrices, and N' denote the total number of critical modes and associated residual modes. Note that N' > N and that $\Omega_{C'}^2$ and $\Phi_{C'}$ have dimension N'×N' and L×N' respectively. The preceding conclusions are modified as follows.

Theorem 1': The N critical modes are completely controllable if and only if the $N' \times (mN')$ augmented matrix

$$Q_{N'} = [\phi_{C'}^T B_A, -\Omega_{C'}^2 \phi_{C'}^T B_A, \ldots, (-\Omega_{C'})^{N-1} \phi_{C'}^T B_A]$$

has rank N'.

Corollary 1': For the N critical modes to be completely controllable, it is sufficient that the augmented influence matrix $\phi_{C'}^T B_A$ has rank N'.

Theorem 3': The N critical modes are completely controllable if and only if

(1) Each row of the influence matrix $\phi_C^T B_A$ is nonzero.

(2) All the rows of the augmented influence matrix $\phi_{C'}^T B_A$ which correspond to a repeated natural frequency are linearly independent.

Corollary 3': For the N critical modes to be completely controllable, the number of actuators used must be at least equal to $\max_{1 \le j \le N'} \mu(\omega_{Cj})$, where $\mu(\omega_{Cj})$ denotes the number of critical and associated residual modes having the same natural frequency ω_{Cj}.

B.6 Magnitude of Control Influence

From the modal Eq. (B-2) it is clear that for applying a given amount of generalized force on mode j, the magnitude $||u||$ of required control input is smaller if the magnitude $||\phi_j^T B_A||$ of the control influence on mode j is larger. Therefore, for controlling mode j, it is desirable (by adjusting the location of the actuators on the structure) to make the control influence on mode j not only nonzero, but also large in magnitude.

Moreover, making the magnitude of control influence on each critical mode as large as possible can avoid requiring excessive input energy or excessively high feedback gains.

B.7 Controllability of Critical Modes Having Nearly Identical Natural Frequencies

Suppose two critical modes (ω_{Cj}, ϕ_{Cj}) and (ω_{Ck}, ϕ_{Ck}) have nearly identical natural frequencies, i.e., $\omega_{Cj} \approx \omega_{Ck}$, but $\omega_{Cj} \neq \omega_{Ck}$. Naturally, one could treat them as having two different natural frequencies, as they are not identical. But it is desirable to ignore the difference.

Assume that both row vectors $\phi_{Cj}^T B_A$ and $\phi_{Ck}^T B_A$ are nonzero, but not linearly independent. Then

$$c_j \phi_{Cj}^T B_A + c_k \phi_{Ck}^T B_A = 0$$

for some nonzero constants c_j and c_k. Similarly, combining the corresponding equations in system (B-2), yields

$$c_j \ddot{\eta}_{Cj} + c_k \ddot{\eta}_{Ck} + \omega_{Cj}^2 c_j \eta_{Cj} + \omega_{Ck}^2 c_k \eta_{Ck} = 0 \qquad \text{(B-18)}$$

Define a combined coordinate

$$\tilde{\eta}_c \triangleq c_j \eta_{Cj} + c_k \eta_{Ck}$$

Then Eq. (B-18) can be rewritten as

$$\ddot{\tilde{\eta}}_c + \omega_{Cj}^2 \tilde{\eta}_c = c_k (\omega_{Cj}^2 - \omega_{Ck}^2) \eta_k$$

$$= c_k (\omega_{Cj}^2 - \omega_{Ck}^2) \frac{1}{\omega_{Ck}} \left[\int_c^t \sin(\omega_{Ck}\tau) \phi_{Ck}^T B_A u(t-\tau) d\tau \right.$$

$$\left. + \eta_{Ck}(0) \omega_{Ck} \cos \omega_{Ck} t + \dot{\eta}_{Ck}(0) \sin \omega_{Ck} t \right]$$

Since $\omega_{Cj} \approx \omega_{Ck}$, the coefficient $c_k(\omega_{Cj}^2 - \omega_{Ck}^2)/\omega_{Ck}$ is negligibly small. Complete controllability of these critical modes is weak and excessive energy is required. Thus, it is better to consider these two modes as having identical frequencies, and to readjust the location of the actuators so that at least $\phi_{Ck}^T B_A$ and $\phi_{Cj}^T B_A$ are linearly independent.

Furthermore, since computational errors are inevitable in natural frequencies, ω_{Cj} and ω_{Ck} might in fact have been identical. Thus, it is desirable that these two modes be treated as having repeated natural frequencies. Condition (2) of Theorem 3 then applies to these two modes, together with any other critical modes having natural frequencies identical or nearly identical to ω_{Cj} or ω_{Ck}.

B.8 Number and Location of Actuators

B.8.1 Determination of Proper Location and Minimum Number

As can be seen from the foregoing analysis, the location and number of the actuators placed on the structure determine whether or not all the critical

modes are controllable. Moreover, improper location may require more actuators than necessary. If the location and the number of the actuators on the structure are adjustable, the following algorithm is proposed for ensuring at least complete controllability of the N critical modes.

Step 1: Initially lay out N actuators. Set $m = N$.

Step 2: Adjust the location of the m actuators so that each critical mode has nontrivial influence from the actuators; namely, $\phi_{Cj}^T B_A \neq 0$ for all $j = 1, \ldots, N$.

Step 3: Readjust the location, if necessary, so that if ω_j is any repeated critical natural frequency with multiplicity $\mu_j \geq 2$, then the corresponding μ_j rows of matrix $\Phi_C^T B_A$ are linearly independent. Otherwise, go back to Step 2.

Step 4: Readjust the location, without violating the criteria in Steps 2 and 3 to increase the magnitudes of control influence on the critical modes:

$$\|\phi_{C1}^T B_A\|^2, \|\phi_{C2}^T B_A\|^2, \ldots, \|\phi_{CN}^T B_A\|^2.$$

Step 5: Readjust the location, without violating the criteria in Steps 2 to 4, to decrease the number of associated residual modes.

Step 6: (If there are no associated residual modes):

If $m = \max_{1 \leq j \leq N} \mu(\omega_{Cj})$, stop;

Otherwise, set $m = \max_{1 \leq j \leq N} \mu(\omega_{Cj})$ and go to Step 7.

(If there are associated residual modes):

If $m = \max_{1 \leq j \leq N'} \mu(\omega_{Cj})$, stop;

Otherwise, set $m = \max_{1 \leq j \leq N'} \mu(\omega_{Cj})$, and go to Step 7.

Step 7: Retain only m actuators, and go back to Step 2.

B.8.2 Decomposition of Modal Modes of Symmetric Structures

Symmetric structures usually exhibit groups of repeated natural frequencies. Structural symmetry is useful to resolve complications with repeated natural frequencies. The mode shapes and modal coordinates may be redefined in such a way that the system (B-2) of modal equations is decomposed into identical (or essentially identical) systems according to the structural symmetry (or the multiplicities in the critical natural frequencies). Each subsystem may be considered as a separate unit, and therefore controlled independently.

The total number of actuators required may remain the same, but the order of complexity in the design of structural control systems may be greatly reduced. The order of each subsystem is much smaller than the overall system, and the control systems designed for one subsystem may be duplicated for the others. Furthermore, since each actuator may concentrate on fewer critical modes (within each subsystem), feedback gains may also be greatly reduced in magnitude.

References

1. Balas, M.J., "Active Control of Flexible Systems," Proceedings of AIAA Symposium on Dynamics and Control of Large Flexible Spacecraft, Blacksburg, VA, June 1977.

2. Canavin, J.R., "The Control of Spacecraft Vibration Using Multivariable Output Feedback," Proceedings of AIAA/AAS Astrodynamics Conference, Palo Alto, CA, Aug. 1978, Paper 78-1419.

3. Simon, J.D., and S.K. Mitter, "A Theory of Modal Control," Information and Control, Vol. 13, pp. 316-353, Oct. 1968.

4. Crossley, T.R. and B. Porter, "Synthesis of Aircraft Modal Control Systems Having Real or Complex Eigenvalues," Aeronautical J. Royal Aeronautical Society, Vol. 73, pp. 138-142, Feb. 1969.

5. Kalman, R.E., "Mathematical Description of Linear Dynamical Systems," J. SIAM Control, Series A, Vol. 1, No. 2, pp. 152-192, 1963.

6. Kalman, R.E., "Contributions to the Theory of Optimal Control," Bol. Soc. Mat. Mexicana, Second Ser., Vol. 5, pp. 102-119, 1960; Proceedings of the Mexico City Conference on Ordinary Differential Equations, 1959.

7. Lee, E.B. and L. Markus, Foundations of Optimal Control Theory, New York:Wiley, 1967.

8. Levine, W.S., T.L. Johnson and M. Athans, "Optimal Limited State Variable Feedback Controllers for Linear Systems," IEEE Transactions on Automatic Control, Vol. AC-16, No. 6, pp. 785-793, Dec. 1971.

9. Kalman, R.E., "On the General Theory of Control Systems," Proc. 1st International Congress on Automatic Control, Moscow, USSR, 1960; London: Butterworth Scientific Publications, Vol. 1, pp. 481-492, 1961.

10. Kalman, R.E., Y.C. Ho, and K.S. Narendra, "Controllability of Linear Dynamical Systems," Contributions to Differential Equations, Vol. 1, No. 2, pp. 189-213, 1962.

11. Zadeh, L.A. and C.A. Desoer, Linear System Theory: The State Space Approach, New York:McGraw-Hill, 1963.

12. Gilbert, E.G., "Controllability and Observability in Multivariable Control Systems," J. SIAM Control, Ser. A. Vol. 2, No. 1, pp. 128-151, 1963.

13. Wonham, W.M., "On Pole Assignment in Multi-Input Controllable Linear Systems," IEEE Transactions on Automatic Control, Vol. AC-12, No. 6, pp. 660-665, Dec. 1967.

APPENDIX C

STATE FEEDBACK CONTROL WITH A
LUENBERGER OBSERVER VIA LINEAR-QUADRATIC REGULATION

C.1 Introduction

The objective of this section is to summarize recent results on the robustness of Linear Quadratic Static Feedback (LQSF) regulators. A review of the LQSF regulator design methodology was given in a previous report [1; Sec. 4.2] and is also contained in Appendix B to Volume 2 of this report.

In Section C.2, the scope of the summary is specified and recent results are subsequently presented. Some comments are given in Section C.3.

C.2 Robustness of LQSF Regulators

C.2.1 Scope

The robustness of LQSF regulators as discussed in this section, specifically refers to the ability of the LQSF regulators to retain stability in the presence of perturbations of the open-loop dynamics. These perturbations include model errors and parameter variations and they may be characterized by nonlinear time-varying changes in the open-loop dynamics.

There has been a considerable amount of published work in the open literature related to the robustness of LQSF regulators [see References 2 - 7]. This section, is intended to be a summary of these recent results. The potential application of the robustness results for LQSF regulators to the control of large space structures will be evaluated in the future.

C.2.2 Gain and Phase Margin of LQSF Regulators

Regarding the robustness properties specific to LQSF regulators, the first significant result is due to Anderson and Moore [2]. They show that single-input LQSF regulators have ±60° phase margin, infinite gain margin, and 50 percent gain reduction tolerance.

Safonov and Athans [3] consider the robustness of a general LQSF regulator:

$$\left.\begin{aligned} \dot{x} &= Ax + Bu, \; x(0) = x_0 \\ \min_u J(x,u) &= \int_0^\infty [x^T Q x + u^T R u] \, dt \\ Q &= Q^T \geq 0, \; R = R^T > 0 \end{aligned}\right\} \qquad \text{(C-1)}$$

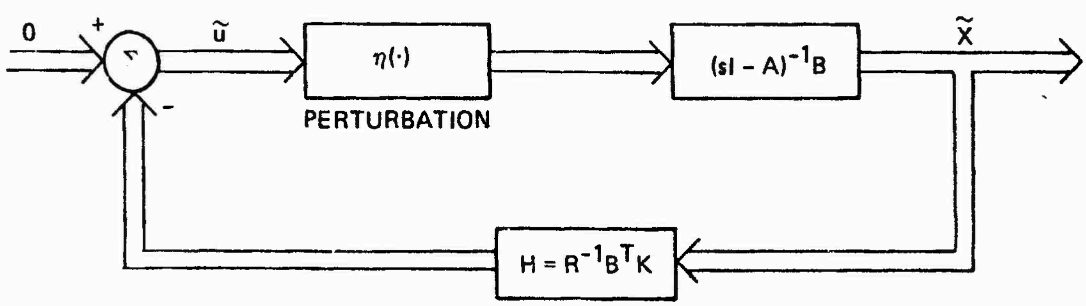
The optimal feedback gain H is given by

$$H = R^{-1}B^TK \tag{C-2}$$

where $K = K^T > 0$ satisfies an algebraic Riccati equation [1; Sec. 4.2.1]. The class of systems considered in reference [3] are perturbed versions of the optimally controlled system, i.e.

$$\left.\begin{aligned} \frac{d}{dt}\tilde{x} &= A\tilde{x} + (B\eta\tilde{u}), \quad \tilde{x}(0) = x_0 \\ \tilde{u} &= -H\tilde{x} \end{aligned}\right\} \tag{C-3}$$

where A, B, x_0, and H are the same as in Eq. (C-1) and (C-2) and η is assumed to be a finite-gain nonanticipative operator with $\eta(0) = 0$. The perturbed LQSF regulator is depicted in the following:



For $\eta(\cdot)$ being a memoryless, time-varying nonlinear operator, a sufficient condition is obtained for the stability of the perturbed LQSF regulator Eq. (C-3). Similarly, a second sufficient stability criterion can be given for $\eta(\cdot)$ being a finite-gain, linear time-invariant operator. However, more interesting results are obtained when special cases are considered.

In particular, consider Eqs. (C-1), (C-2), and (C-3) but specify that

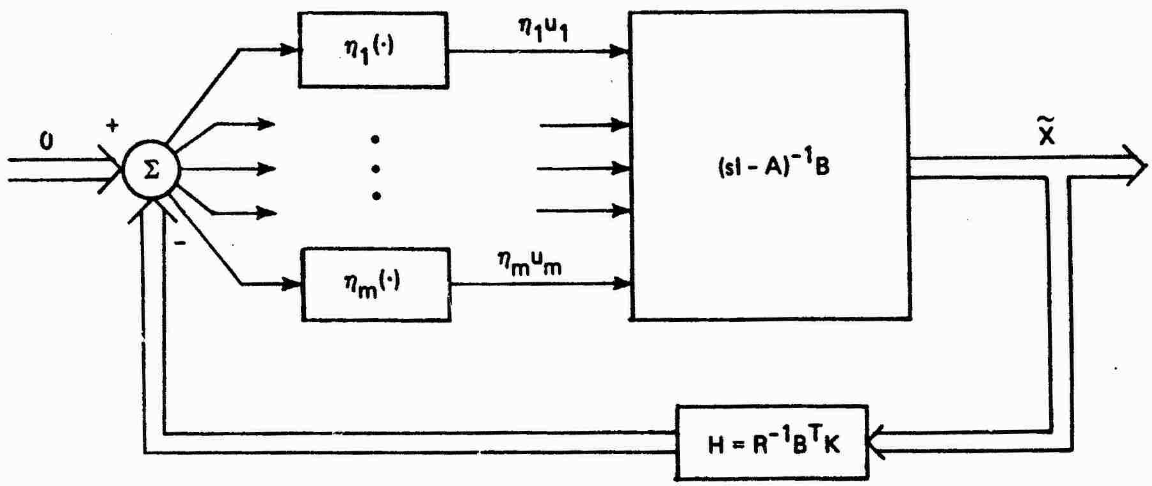
$$Q > 0$$

$$R = \text{a diagonal matrix} > 0$$

and that the perturbation η satisfies

$$\eta u = \begin{bmatrix} \eta_1 u_1 \\ \cdot \\ \cdot \\ \cdot \\ \eta_m u_m \end{bmatrix}$$

so that the perturbations in the various feedback loops are noninteracting. This particular perturbation is illustrated in the following figure:



Under these conditions, the perturbed system remains asymptotically stable in the large, if each of the perturbations η_i is memoryless with $(\eta_i u_i)(t) = f_i(u_i(t),t)$, and if for some $k < \infty$, some $\beta \geq 0$, and all $t \in [0,\infty)$

$$f_i(0,t) = 0$$

$$k \geq \frac{1}{u_i} f_i(u,t) \geq \frac{\beta + 1}{2}, \quad \text{for all } u_i \neq 0$$

In particular, this result implies that either of the following changes leaves an LQSF regulator asymptotically stable in the large:

(1) A phase shift of less than or equal to 60° in the respective feedback loops of each of the controls u_i.

(2) The insertion of linear constant gains (a_i) with $a_i \geq 1/2$ into the feedback loops of the respective controls u_i.

C.2.3 Robustness of LQSF with a Prescribed Degree of Stability

Patel et al [4] consider the robustness of LQSF regulators with a prescribed degree of stability [1, Sec. 4.2.7.3; 2] and, by Lyapunov's method,

establish quantitative bounds on perturbations in the system such that the closed-loop system remains stable. These bounds are obtained for the general case of nonlinear, time-varying perturbations and are determined by the prescribed degree of stability (α), and the maximum and the minimum eigenvalues of two symmetric matrices. Since these two symmetric matrices consist of weighting matrices in the performance index, a designer can select appropriate weighting matrices to attain a robust design.

C.2.4 Gain and Phase Margin of LQSF with Kalman Filter

The robustness properties of LQSF regulators obtained by Safonov and Athans (see Section C.2) hold independent of plant dynamics or performance index selection. They are global and guaranteed. However, recently, Doyle [5] showed by a counterexample that a standard two-state LQG control design results in a closed-loop regulator that has arbitrarily small gain margin. Therefore, there is no guaranteed gain margin for LQSF regulators with Kalman filters.

Similarly, it has been stated that no guaranteed properties hold for LQSF regulators with observers [6].

It is important to note, however, that there may exist approaches to improve the robustness of LQSF regulators with either a Kalman filter or Luenburger observer, even though they do not have guaranteed robustness properties.

C.2.5 Improving the Robustness of LQSF Regulators with either Kalman Filter or Luenburger Observer

The robustness properties of LQSF regulators with filters or observers need to be separately evaluated for each design, since no guaranteed robustness properties hold in these cases. Doyle and Stein [6] present a design adjustment procedure to improve the robustness of such systems. They show that in general the approach of "speeding-up" filter or observer dynamics will not work. To increase the robustness, their procedures are to drive some filter or observer poles toward stable plant zeros and the rest toward infinity. When the procedures are applied to an illustrative example of an LQSF regulator with a Kalman filter, they are equivalent to a method of trading off between noise rejection and margin recovery. It is stated [6] that full-state robustness (see Section C.2) can be recovered asymptotically, if the plant is minimum phase and correct procedures are followed.

C.2.6 Robustness of Discrete-Time LQSF Regulators

Safonov [7] considers the robustness of discrete-time LQSF regulators and obtains the following results: discrete-time LQSF regulators have neither the 6-dB gain reduction, nor the +∞ dB gain increase, nor the ±60° phase uncertainty tolerance of their continuous-time counter parts. Instead, the gains (a_i) in each control channel must lie between the following limits

$$\frac{1}{2} < \frac{1}{1+b_i} \leq a_i \leq \frac{1}{1-b_i} \leq \infty \quad (b_i < 1)$$

and phase uncertainties (ϕ_i) must be bounded by

$$\phi_i \leq 2 \sin^{-1} \frac{b_i}{2} < 60°$$

Detailed expressions for these discrete gain and phase margins as well as tolerance bounds for more general types of nonlinear, time-varying, and dynamic uncertainties can be found in [7].

C.3 Comments

The robustness properties of LQSF regulators (Section C.2) were discussed with respect to perturbations such as parameter errors, parameter variations and gain variations. It should be noted that throughout the discussions of this section the order of the model of the open-loop system was assumed to be correct. In particular, the robustness of LQSF regulators against model truncation errors should be of interest to the control of large space structures.

The results presented in Reference [8] appear to imply that LQSF regulators without filters or observers are robust against model truncation errors due to the absence of observation spillover. LQSF regulators with filters or observers do not seem to have robustness against model truncation errors due to both observation and control spillover. Design procedures in Reference [5] should be evaluated for the potential application in improving the robustness of LQSF regulators with filters or observers against truncation errors.

References

1. Passive and Active Suppression of Vibration Response in Precision Structures, State-of-the Art Assessment, CSDL Report R-1138, Volume II, Feb. 1978.

2. B.D.O. Anderson and J.B. Moore, Linear Optimal Control, Englewood Cliffs, N.J., Prentice-Hall, 1971.

3. M.G. Safonov and M. Athans, "Gain and Phase Margin for Multiloop LQG Regulators", IEEE Trans. Automatic Control, Vol. AC-22, pp. 173-178, April 1977.

4. J.C. Doyle, "Guaranteed Margins for LQG Regulators", IEEE Trans. Automatic Control, Vol. AC-23, pp. 756-757, June 1978.

5. J.C. Doyle and G. Stein, "Robustness with Observers", Proc. 1978 IEEE Conf. on Decision and Control, San Diego, California, January 10-12, 1979, pp. 1-6.

6. R.A. Patel, M. Toda and S. Sridhar, "Robustness of Linear Quadratic State Feedback Designs in the Presence of System Uncertainty", IEEE Trans. Automatic Control, Vol. AC-22, pp. 945-949, December 1977.

7. M.G. Safonov, "Robustness and Stability Aspects of Stochastic Multivariable Feedback System Design", Ph.D. Thesis, MIT, Sept. 1977.

8. M.J. Balas, "Feedback Control of Flexible Systems", IEEE Trans. Automatic Control, Vol. AC-23, pp. 673-679, Aug. 1978.

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